DISCRETE FOURIER SERIES & FOURIER TRANSFORMS

Discrete Fourier Transform

The Discrete Fourier transform (DFT) is a powerful computation tool which allows to evaluate the Fourier transform $X(e^{j\omega})$ on a digital computer or specially designed hardware.

Unlike DTFT, which is defined for finite and infinite sequences, DFT is defined only for sequences of finite length. Since $X(e^{j\omega})$ is continous and periodic, DFT is obtained by sampling one period of the Fourier transform at a finite no. of frequency points.

Discrete Fourier Series

The Discrete Fourier series (DFS) is represented by

$$X_p(k) = \sum_{n=0}^{N-1} x_p(n) e^{-j2\pi kn/N}$$

The Inverse Discrete Fourier series (IDFS) is represented by

$$x_p(n) = \frac{1}{N} \sum_{n=0}^{N-1} X_p(k) e^{j2\pi kn/N} \quad ; n = 0, 1, \ldots N-1$$

where $X_p(k), k = 0, 1, \ldots N-1$ are called Discrete Fourier series coefficients

→ Properties of Discrete Fourier series

1. Linearity

Consider two periodic sequences $x_{1p}(n)$, $x_{2p}(n)$ both with period N, such that

$$DFS[x_{1p}(n)] = X_{1p}(k)$$

$$DFS[x_{2p}(n)] = X_{2p}(k)$$

then

$$DFS[a_1 x_{1p}(n) + a_2 x_{2p}(n)] = a_1 X_{1p}(k) + a_2 X_{2p}(k)$$

2. Time shifting property

If $x_p(n)$ is a periodic sequence with period N and

$$DFS[x_p(n)] = X_p(k)$$

then $DFS[x_p(n-m)] = e^{-j(2\pi/N)mk} X_p(k)$

where $x_p(n-m)$ is a shifted version of $x_p(n)$

3. Symmetry Property

If $DFS[x_p(n)] = X_p(k)$ then

$$DFS[x_p^*(n)] = X_p^*(-k)$$

$$DFS[x_p^*(-n)] = X_p^*(k)$$

and

$$DFS\{Re[x_p(n)]\} = DFS\left[\frac{x_p(n) + x_p^*(n)}{2}\right] = \frac{1}{2}[X_p(k) + X_p^*(-k)]$$

$$= X_{pe}(k)$$

$$DFS\{j\,Im[x_p(n)]\} = DFS\left[\frac{x_p(n) - x_p^*(n)}{2}\right]$$

$$= \frac{1}{2}[X_p(k) - X_p^*(-k)] = X_{po}(k)$$

We can write $x_p(n)$ as

$$x_p(n) = x_{pe}(n) + x_{po}(n)$$

where $x_{pe}(n) = \frac{1}{2}[x_p(n) + x_p^*(-n)]$

$$x_{po}(n) = \frac{1}{2}[x_p(n) - x_p^*(-n)]$$

then

$$DFS[x_{pe}(n)] = DFS\left\{\frac{1}{2}[x_p(n) + x_p^*(-n)]\right\}$$

$$= \frac{1}{2}[X_p(k) + X_p^*(k)] = Re\{X_p(k)\}$$

$$DFS[x_{po}(n)] = DFS\left\{\frac{1}{2}[x_p(n) - x_p^*(-n)]\right\}$$

$$= \frac{1}{2}[X_p(k) - X_p^*(k)] = j\,Im\{X_p(k)\}$$

4. Periodic Convolution

Let $x_{1p}(n)$ and $x_{2p}(n)$ be two periodic sequences with period 'N' with

$$DFS[x_{1p}(n)] = X_{1p}(k)$$

$$DFS[x_{2p}(n)] = X_{2p}(k)$$

If $X_{3p}(k) = X_{1p}(k)\,X_{2p}(k)$ then the periodic sequences $x_{3p}(n)$ with Fourier series co-efficients $X_{3p}(k)$ is

$$x_{3p}(n) = \sum_{m=0}^{N-1} x_{1p}(m)\,x_{2p}(n-m)$$

In summary

$$DFS\left[\sum_{m=0}^{N-1} x_{1p}(m)\,x_{2p}(n-m)\right] = X_{1p}(k)\,X_{2p}(k)$$

Let us define a term

$$W_N = e^{-j2\pi/N}$$

which is known as twiddle factor

→ Find the DFT of a sequence $x(n) = \{1, 1, 0, 0\}$ and find the IDFT of $Y(K) = \{1, 0, 1, 0\}$.

Sol. Let us assume $N = L = 4$

We have $X(K) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi nk/N}$ $K = 0, 1, \ldots N-1$

$\underline{k=0}$. $X(0) = \sum_{n=0}^{3} x(n)e^{0} = x(1) + x(2) + x(3) + x(4)$

$= 1 + 1 + 0 + 0 = 2$

$\underline{K=1}$

$X(1) = \sum_{n=0}^{3} x(n) e^{-j2\pi n/4} = \sum_{n=0}^{3} x(n) e^{-jn\pi/2}$

$= x(0) + x(1)e^{-j\pi/2} + x(2)e^{-j\pi} + x(3)e^{-j3\pi/2}$

$= 1 + 1(\cos\frac{\pi}{2} - j\sin\frac{\pi}{2}) + 0 + 0$

$= 1 + 0 - j = 1 - j$

$\underline{K=2}$

$X(2) = \sum_{n=0}^{3} x(n) e^{-j\cancel{2}\pi n 3/\cancel{4}2} = \sum_{n=0}^{3} x(n) e^{-j3\pi n/2}$

$= x(0) + x(1)e^{-j3\pi/2} + x(2)e^{-j3\pi} + x(3)e^{-j9\pi/2}$

$= 1 + \cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2}$.

$= 1 + 0 - j(-1) = 1 + j$

$\cos\frac{3\pi}{2} = \cos(\pi + \frac{\pi}{2}) = 0$

$\sin\frac{3\pi}{2} = \sin(\pi + \frac{\pi}{2}) = -1$

$\underline{K=3}$.

$X(3) = \sum_{n=0}^{3} x(n) e^{-j\cancel{2}\pi n\cancel{3}/\cancel{4}} = \sum_{n=0}^{3} x(n) e^{-jn\pi}$

$= x(0) + x(1)e^{-j\pi} + x(2)e^{-j2\pi} + x(3)e^{-j3\pi}$

$= 1 + \cos\pi - j\sin\pi = 1 + -1 + 0 = \underline{0}$

$\cos(\frac{\pi}{2} + \frac{\pi}{2}) = -\cancel{\cos}\sin\frac{\pi}{2} = 0$

$\sin(\frac{\pi}{2} + \frac{\pi}{2}) =$

$\therefore X(K) = \{2, 1-j, 0, 1+j\}$

The IDFT of $Y(K) = \{1, 0, 1, 0\}$ is ...

$$y(n) = \frac{1}{N}\sum_{K=0}^{N-1} Y(K) e^{j2\pi nk/N} \qquad n = 0, 1, 2 \ldots N-1$$

$\underline{\underline{n=0}}$

$$y(0) = \frac{1}{4}\sum_{K=0}^{3} Y(K) e^{0} = \frac{1}{4}\left[Y(0) + Y(1) + Y(2) + Y(3)\right]$$

$$= \frac{1}{4}\left[1+0+1+0\right] = \frac{2}{4} = \frac{1}{2} = 0.5.$$

$\underline{\underline{n=1}}$

$$y(1) = \frac{1}{4}\sum_{K=0}^{3} Y(K) e^{+j2\pi K \cdot /4} = \frac{1}{4}\sum_{K=0}^{3} Y(K) e^{jK\pi/2}$$

$$= \frac{1}{4}\left[Y(0) + Y(1) e^{j\pi/2} + Y(2) e^{j\pi} + Y(3) e^{j3\pi/2}\right]$$

$$= \frac{1}{4}\left[1 + 0 + (\cos\pi + j\sin\pi) + 0\right] = \frac{1}{4}\left[1+0-1+0\right] = 0$$

$\underline{\underline{n=2}}$

$$y(2) = \frac{1}{4}\sum_{K=0}^{3} Y(K) e^{j2\pi \cdot 2K/4} = \frac{1}{4}\sum_{K=0}^{3} Y(k) e^{j\pi K}$$

$$= \frac{1}{4}\left[Y(0) + Y(1) e^{j\pi} + Y(2) e^{j2\pi} + Y(3) e^{j3\pi}\right]$$

$$= \frac{1}{4}\left[1 + 0 + \cos 2\pi + j\sin 2\pi\right] = \frac{1}{4}\left[1+0+1+0\right] = \frac{2}{4} = \underline{\underline{0.5}}$$

$\cos(\pi+\pi)$

$-\cos(\frac{\pi}{2}+\frac{\pi}{2})$

$\sin\frac{\pi}{2} = 1$

$\underline{\underline{n=3}}$

$$y(3) = \frac{1}{4}\sum_{K=0}^{3} Y(k) e^{j\cancel{2}\pi 3k/\cancel{4}2} = \frac{1}{4}\sum_{K=0}^{3} Y(K) e^{j3\pi K/2}$$

$$= \frac{1}{4}\left[Y(0) + Y(1) e^{j3\pi/2} + Y(2) e^{j3\pi} + Y(3) e^{j9\pi/2}\right]$$

$$= \frac{1}{4}\left[1 + 0 + 1(\cos 3\pi + j\sin 3\pi) + 0\right] = \frac{1}{4}\left[1 + 0 + (-1) + 0\right] = 0.$$

$\therefore y(n) = \{0.5, 0, 0.5, 0\}$

→ Find the DFT of a sequence $x(n) = 1$ for $0 \le n \le 2$
$= 0$ otherwise

for (i) $N = 4$ (ii) $N = 8$. Plot $|X(k)|$ and $\angle X(k)$. Comment on the result.

Sol. Given $L = 3$. For $N = 4$, the periodic extension of $x(n)$ can be obtained by adding one zero (i.e. '$N-L$' zeros) ; Given $x(n) = \{1, 1, 1, 0\}$.

We have

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} \quad ; k = 0, 1, \ldots N-1$$

$\underline{k=0}$

$$X(0) = \sum_{n=0}^{3} x(n) e^{0} = x(0) + x(1) + x(2) + x(3) = 1+1+1+0 = 3.$$

Therefore $|X(0)| = 3$, $\angle X(0) = 0$

$\underline{k=1}$

$$X(1) = \sum_{n=0}^{3} x(n) e^{-j2\pi n/4} = \sum_{n=0}^{3} x(n) e^{-jn\pi/2}$$

$$= [x(0) + x(1) e^{-j\pi/2} + x(2) e^{-j\pi} + x(3) e^{-j3\pi/2}]$$

$$= [1 + \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} + \cos\pi - j\sin\pi]$$

$$= 1 + 0 - j - 1 = -j$$

$|X(1)| = 1$, $\angle X(1) = -\pi/2$.

$\cos(\pi + \pi) = \cos(\frac{\pi}{2}) \ldots$

$\sin(\pi + \pi) = \ldots \sin\pi$

$\underline{k=2}$

$$X(2) = \sum_{n=0}^{3} x(n) e^{-j2\pi n \cdot 2/4} = \sum_{n=0}^{3} x(n) e^{-jn\pi}$$

$$= x(0) + x(1) e^{-j\pi} + x(2) e^{-j2\pi} + x(3) e^{-j3\pi}$$

$$= 1 + \cos\pi - j\sin\pi + \cos 2\pi - j\sin 2\pi = 1 - 1 - j(0) + 1 = 1$$

$|X(2)| = 1, \quad \angle X(2) = 0$

For $k=3$

$$X(3) = \sum_{n=0}^{3} x(n) e^{-j2\pi nk/N} = \sum_{n=0}^{3} x(n) e^{-j3\pi n/2}$$

$\cos(\pi + \frac{\pi}{2}) = 0$

$$= x(0) + x(1)e^{-j3\pi/2} + x(2)e^{-j3\pi} + x(3)e^{-j9\pi/2}$$

$\cos(2\pi + \pi) = \cos(\pi)$

$\sin(2\pi + \pi) = \sin(\pi)$

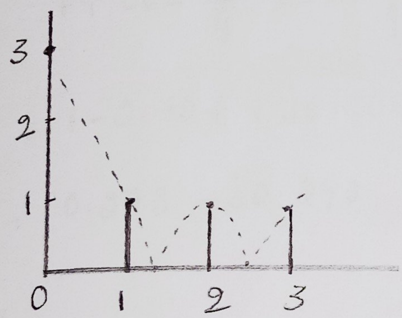
$$= 1 + \cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2} + \cos 3\pi - j\sin 3\pi$$

$$= 1 + 0 - j(-1) + (-1) - j(0) = 1 + j - 1 = j$$

$|X(3)| = 1 \quad ; \quad \angle X(3) = \frac{\pi}{2}.$

$\therefore |X(k)| = \{3, 1, 1, 1\} \quad ; \quad \angle X(k) = \{0, \frac{-\pi}{2}, 0, \frac{\pi}{2}\}$

The plot of $|X(k)|$ & $\angle X(k)$ for $N=4$.



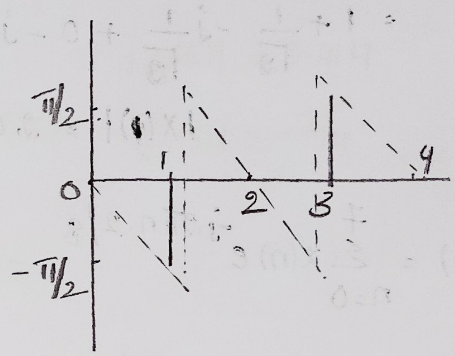
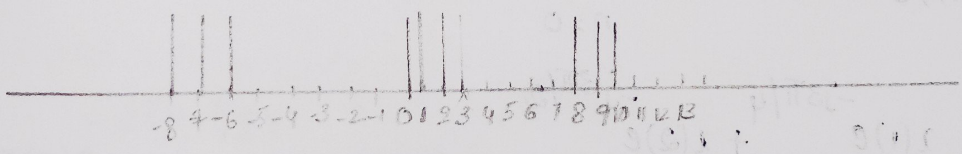


Fig1 Frequency response of $x(n)$ for $N=4$.

For $N=8$, the periodic extension of $x(n)$ shown in below figure can be obtained by adding $5(N-L = 8-3)$ zeros. i.e

$x(0) = x(1) = x(2) = 1 \quad ; \quad x(n) = 0 \quad$ for $3 \leq n \leq 7$



We know that $X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi nk/N}$.

$\underline{\underline{k=0}}$

$$X(0) = \sum_{n=0}^{7} x(n) e^{0} = x(0)+x(1)+x(2)+x(3)+x(4)+x(5)+x(6)+x(7)$$

$$= 1+1+1+0+0+0+0+0 = 3$$

$|X(0)| = 3$; $\angle X(0) = 0$

$\underline{\underline{k=1}}$

$$X(1) = \sum_{n=0}^{7} x(n) e^{-j2\pi n/8} = \sum_{n=0}^{7} x(n) e^{-jn\pi/4}$$

$$= x(0) + x(1)e^{-j\pi/4} + x(2)e^{-j\pi/2} + x(3)e^{-j3\pi/4} + x(4)e^{-j\pi} + x(5)e^{-j5\pi/4}$$

$$+ x(6)e^{-j3\pi/2} + x(7)e^{-j7\pi/4}$$

$$= 1 + \cos\frac{\pi}{4} - j\sin\frac{\pi}{4} + \cos\frac{\pi}{2} - j\sin\frac{\pi}{2}$$

$$= 1 + \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} + 0 - j = 1 + \frac{1}{\sqrt{2}} - j\left(1+\frac{1}{\sqrt{2}}\right) = 1.707 - j1.707.$$

$|X(1)| = 2.414$ $\angle X(1) = -\pi/4$.

$\underline{\underline{k=2}}$

$$X(2) = \sum_{n=0}^{7} x(n) e^{-j2\pi n\cdot 2/8} = \sum_{n=0}^{7} x(n) e^{-jn\pi/2}$$

$$= x(0) + x(1)e^{-j\pi/2} + x(2)e^{-j\pi}$$

$$= 1 + \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} + \cos\pi - j\sin\pi$$

$$= 1 - j - 1 - 0 = -j$$

$|X(2)| = 1$; $\angle X(2) = -\pi/2$

$\underline{\underline{k=3}}$

$$X(3) = \sum_{n=0}^{7} x(n) e^{-j2\pi n\cdot 3/8} = \sum_{n=0}^{7} x(n) e^{-j3\pi n/4}$$

$$= x(0) + x(1)e^{-j3\pi/4} + x(2)e^{-j3\pi/2}$$

$$= 1 + \cos\frac{3\pi}{4} - j\sin\frac{3\pi}{4} + \cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2} = 1 + (-0.707) - j0.707 + 0 - j(-1)$$

$= 1 - 0.707 - j0.707 + j$ $\qquad |X(3)| = 0.414$

$= 0.293 + j0.293$ $\qquad \angle X(3) = \frac{\pi}{4}$

$\underline{\underline{k=4}}$

$X(4) = \sum_{n=0}^{7} x(n) e^{-j2\pi n \cdot 4/8} = \sum_{n=0}^{7} x(n) e^{-jn\pi}$

$= x(0) + x(1) e^{-j\pi} + x(2) e^{-j2\pi}$

$= 1 + \cos\pi - j\sin\pi + \cos 2\pi - j\sin 2\pi$ $\qquad |X(4)| = 1$

$= 1 - 1 + 0 + 1 - 0 \quad = 1$ $\qquad \angle X(4) = 0$

$\underline{\underline{k=5}}$

$X(5) = \sum_{n=0}^{7} x(n) e^{-j2\pi n \cdot 5/8} = \sum_{n=0}^{7} x(n) e^{-j5\pi n/4}$

$= x(0) + x(1) e^{-j5\pi/4} + x(2) e^{-j5\pi/2}$

$= 1 + \cos\frac{5\pi}{4} - j\sin\frac{5\pi}{4} + \cos\frac{5\pi}{2} - j\sin\frac{5\pi}{2}$ $\qquad |X(5)| = 0.414$

$= 1 - 0.707 + j0.707 + 0 - j$ $\qquad \angle X(5) = -\pi/4$

$= 0.293 - j0.293$

$\underline{\underline{k=6}}$

$X(6) = \sum_{n=0}^{7} x(n) e^{-j2\pi n \cdot 6/8} = \sum_{n=0}^{7} x(n) e^{-j3n\pi/2}$

$= x(0) + x(1) e^{-j3\pi/2} + x(2) e^{-j3\pi}$ $\qquad |X(6)| = 1$

$= 1 + \cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2} + \cos 3\pi - j\sin 3\pi$ $\qquad \angle X(6) = \pi/2$

$= 1 + 0 + j - 1 - j(0) = j$

$\underline{\underline{k=7}}$

$X(7) = \sum_{n=0}^{7} x(n) e^{-j2\pi n \cdot 7/8} = \sum_{n=0}^{7} x(n) e^{-j7\pi n/4}$

$$X(7) = \sum_{n=0}^{7} x(n) e^{-j7\pi n/4}$$

$$= x(0) + x(1)e^{-j7\pi/4} + x(2)e^{-j7\pi/2}$$

$$= 1 + \cos\frac{7\pi}{4} - j\sin\frac{7\pi}{4} + \cos\frac{7\pi}{2} - j\sin\frac{7\pi}{2}$$

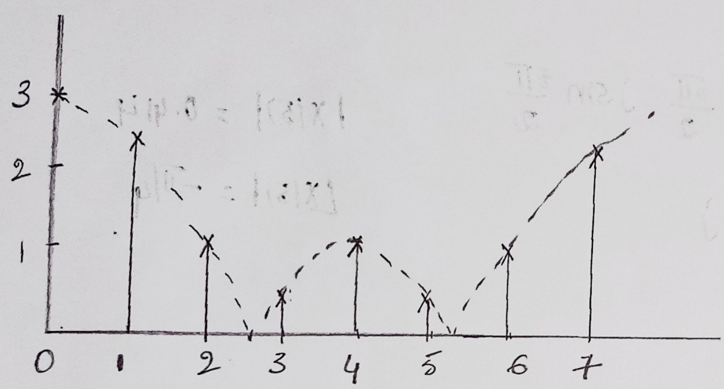
$$= 1 + 0.707 + j0.707 + 0 + j$$

$$= 1.707 + j1.707$$

$|X(7)| = 2.414$; $\angle X(7) = \pi/4$

$\therefore |X(K)| = \{3, 2.414, 1, 0.414, 1, 0.414, 1, 2.414\}$

$\angle X(K) = \{0, -\pi/4, -\pi/2, \pi/4, 0, -\pi/4, \pi/2, \pi/4\}$.



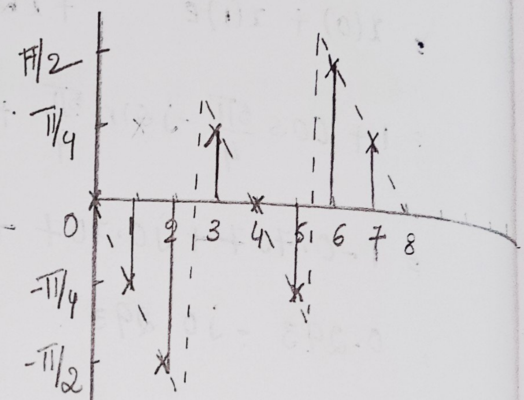


Fig2: Frequency response of $x(n)$ for $N=8$.

From fig 1, we can observe that with $N=4$, it is difficult to extrapolate the entire frequency spectrum. For low values of N, the spacing between successive samples is high, which results in poor resolution.

On the other hand, when $N=8$ from fig 2 we can observe that it is possible to extrapolate the frequency of spectrum.

That is zero padding gives a high density spectrum and provides a better displayed version for plotting.

Properties of DFT

1. Periodicity: If a sequence $x(n)$ is periodic with periodicity of 'N' samples, then N-point DFT of the sequence, $X(k)$ is also periodic with periodicity of N samples.

Hence, if $x(n)$ and $X(k)$ are an N-point DFT pair, then

$$x(n+N) = x(n) \quad \text{for all } n$$

$$X(k+N) = X(k) \quad \text{for all } k$$

Proof: By definition of DFT, the $(k+N)^{th}$ co-efficient of $X(k)$ is given by

$$X(k+N) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi n(k+N)/N}$$

$$= \sum_{n=0}^{N-1} x(n) e^{-j2\pi nk/N} \cdot e^{-j2\pi nN/N}$$

$$= \sum_{n=0}^{N-1} x(n) e^{-j2\pi nk/N} \cdot e^{-j2\pi n}$$

$$= \sum_{n=0}^{N-1} x(n) e^{-j2\pi nk/N} \qquad \left[\because e^{-j2\pi n} = 1\right]$$

$$= X(k).$$

2. Linearity:

If $x_1(n)$ and $x_2(n)$ are two finite duration sequences and if

$$DFT\{x_1(n)\} = X_1(k)$$

and $$DFT\{x_2(n)\} = X_2(k)$$

Then for any real valued or complex valued constants 'a' & 'b',

$$DFT\{a x_1(n) + b x_2(n)\} = a X_1(k) + b X_2(k)$$

Proof: $DFT\{ax_1(n)+bx_2(n)\} = \sum_{n=0}^{N-1}[ax_1(n)+bx_2(n)]e^{-j2\pi nk/N}$

$$= a\sum_{n=0}^{N-1} x_1(n)e^{-j2\pi nk/N} + b\sum_{n=0}^{N-1} x_2(n)e^{-j2\pi nk/N}$$

$$= aX_1(K) + bX_2(K)$$

3. DFT of Even and odd Sequences

The DFT of an even sequence is purely real and the DFT of an odd sequence is purely imaginary.

Therefore, DFT can be evaluated using cosine and sine transforms for even and odd sequences respectively

For even sequence, $X(K) = \sum_{n=0}^{N-1} x(n)\cos\left(\frac{2\pi nk}{N}\right)$

For odd sequence, $X(K) = \sum_{n=0}^{N-1} x(n)\sin\left(\frac{2\pi nk}{N}\right)$

4. Time Reversal of the sequence

The time reversal of an N-point sequence $x(n)$ is obtained by wrapping the sequence $x(n)$ around the circle in the clockwise direction. It is denoted as $[x(-n), \text{Mod } N]$ and

$$x[(-n), \text{mod } N] = x(N-m), \quad 0 \leq n \leq N-1$$

If $DFT\{x(n)\} = X(K)$ then

$$DFT\{x(-n), \text{mod } N\} = DFT\{x(N-n)\}$$

$$= X[(-K), \text{mod } N] = X(N-K)$$

Proof:

$$DFT\{x(N-n)\} = \sum_{n=0}^{N-1} x(N-n) e^{-j2\pi nk/N}$$

changing index from n to m, where $m = N-n$,

then $n = N-m$

Now, $$DFT\{x(N-n)\} = \sum_{m=0}^{N-1} x(m) e^{-j2\pi (N-m)k/N}$$

$$= \sum_{m=0}^{N-1} x(m) e^{-j2\pi Nk/N} \cdot e^{+j2\pi mk/N}$$

$$= \sum_{m=0}^{N-1} x(m) e^{-j2\pi k} \cdot e^{j2\pi mk/N}$$

$$= \sum_{m=0}^{N-1} x(m) e^{j2\pi mk/N} \qquad [\because e^{-j2\pi k} = 1 \text{ for } k = 0,1,2,3 \ldots]$$

$$= \sum_{m=0}^{N-1} x(m) e^{j2\pi mk/N} \cdot e^{-j2\pi m}$$

$$= \sum_{m=0}^{N-1} x(m) e^{-j2\pi m(1-k/N)}$$

$$= \sum_{m=0}^{N-1} x(m) e^{-j2\pi m(N-k)/N}$$

$$DFT\{x(N-n)\} = X(N-k)$$

5. Circular Frequency shift

If $DFT\{x(n)\} = X(k)$

Then $DFT\{x(n) e^{j2\pi ln/N}\} = X[(k-l), (\text{mod } N)]$

Proof: $\text{DFT}\{x(n)e^{j2\pi ln/N}\} = \sum_{n=0}^{N-1} x(n)e^{j2\pi ln/N} \cdot e^{-j2\pi kn/N}$

$$= \sum_{n=0}^{N-1} x(n)e^{-j2\pi n(k-l)/N}$$

$$= \sum_{n=0}^{N-1} x(n)e^{-j2\pi n(k-l)/N} \cdot e^{-j2\pi nN/N}$$

$$= \sum_{n=0}^{N-l} x(n)e^{-j2\pi n(N+k-l)/N}$$

$$= X(N+k-l)$$

$$= X[(k-l), (\text{mod } N)]$$

→ Parseval's Theorem

Parseval's theorem says that the DFT is an energy-conserving transformation and allows us to find the signal energy either from the signal or its spectrum.

This implies that the sum of squares of the signal samples is related to the sum of squares of the magnitude of the DFT samples.

If $\text{DFT}[x_1(n)] = X_1(k)$

and $\text{DFT}\{x_2(n)\} = X_2(k)$

Then $\sum_{n=0}^{N-1} x_1(n)x_2^*(n) = \frac{1}{N}\sum_{k=0}^{N-1} X_1(k)X_2^*(k)$

Properties of the DFT

Property	Time-Domain	Frequency domain
1. Periodicity	$x(n) = x(n+N)$	$X(K) = X(k+N)$
2. Linearity	$a_1 x_1(n) + a_2 x_2(n)$	$a_1 X_1(K) + a_2 X_2(K)$
3. Time reversal	$x(N-n)$	$X(N-K)$
4. Circular time shift	$x((n-l))_N$	$X(k) e^{-j2\pi kl/N}$
5. Circular frequency shift	$x(n) e^{j2\pi ln/N}$	$X((K-l))_N$
6. Circular convolution	$x_1(n)$ (N) $x_2(n)$	$X_1(k) X_2(k)$
7. Circular correlation	$x_1(n)$ (N) $y^*(-n)$	$X(k) Y^*(k)$
8. Multiplication of two sequences	$x_1(n) x_2(n)$	$\frac{1}{N}[X(k)$ (N) $X_2(k)]$
9. Complex conjugate	$x^*(n)$	$X^*(N-k)$
10. Parseval's Theorem	$\sum_{n=0}^{N-1} x(n) y^*(n)$	$\frac{1}{N} \sum_{k=0}^{N-1} X(k) Y^*(k)$

Complex conjugate property

If $DFT\{x(n)\} = X(k)$

then $DFT[x^*(n)] = X^*(N-k) = X^*((-k))_N$

Proof:

$$DFT[x^*(n)] = \sum_{n=0}^{N-1} x^*(n) e^{-j2\pi nk/N}$$

$$= \left[\sum_{n=0}^{N-1} x(n) e^{j2\pi nk/N} \right]^*$$

$$= \left[\sum_{n=0}^{N-1} x(n) e^{-j2\pi n(N-k)/N} \right]^*$$

$$DFT[x^*(n)] = X^*(N-k)$$

IIIy $DFT[x^*(N-n)] = X^*(k)$

→ Consider the length - 6 sequence defined for $0 \le n \le 6$

$x(n) = \{1, -2, 3, 0, -1, 1\}$ with a 8-point DFT $X(k)$. Evaluate the following functions of $X(k)$ without computing DFT.

(a) $X(0)$, (b) $X(3)$, (c) $\sum_{k=0}^{5} X(k)$ (d) $\sum_{K=0}^{5} |X(k)|^2$

Sol.

We know that $X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi nk/N}$, $k = 0, 1 \ldots N-1$.

$$\therefore X(0) = \sum_{n=0}^{N-1} x(n) e^{0} = \sum_{n=0}^{5} x(n)$$

$$= x(0) + x(1) + x(2) + x(3) + x(4) + x(5)$$

$$= 1 - 2 + 3 + 0 - 1 + 1 = 2$$

(b) $X(3)$

$\cos(4\pi+\theta)$

$\cos$

$$X(3) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi n \cdot \cancel{3}/\cancel{6}} = \sum_{n=0}^{N-1} x(n) e^{-jn\pi}$$

$$= x(0) + x(1) e^{-j\pi} + x(2) e^{-j2\pi} + x(3) e^{-j3\pi} + x(4) e^{-j4\pi} + x(5) e^{-j5\pi}$$

$$= 1 + (-2)[-1 - j0] + 3[1 - j(0)] + 0 - 1[1] + 1[-1]$$

$$= 1 + 2 + 3 - 1 - 1 = \underline{4}$$

(c) We know that $x(n) = \frac{1}{N} \sum_{n=0}^{N-1} X(k) e^{j2\pi nk/N}$, $k = 0, 1, \ldots N-1$.

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j2\pi nk/N}$$

at $n = 0$

$$x(0) = \frac{1}{N} \sum_{k=0}^{5} X(k) \Rightarrow \sum_{k=0}^{5} X(k) = N x(0)$$

$$= 6(1) = 6.$$

(d) From Parseval's theorem

$$\sum_{k=0}^{5} |X(k)|^2 = N \sum_{n=0}^{5} |x(n)|^2 = 6[(1)^2 + (-2)^2 + (3)^2 + (0)^2 + (-1)^2 + (1)^2]$$

$$= 6[1 + 4 + 9 + 0 + 1 + 1] = 6 \times 16 = \underline{96}$$

If the DFT $\{x(n)\} = X(K) = \{4, -j2, 0, j2\}$ using properties of DFT, find

(a) DFT of $x(n-2)$

(b) DFT of $x(-n)$

(c) DFT of $x^*(n)$

(d) DFT of $x^2(n)$

(e) DFT of $x(n) \circledast x(n)$

(f) signal energy

Sol (a) Using the time shifting property of DFT, We have $N = 4$

$$DFT\{x(n-2)\} = e^{-j2\pi 2K/4} X(K)$$

$$= e^{-j\pi K} X(K)$$

$$= \{X(0),\ X(1)e^{-j\pi},\ X(2)e^{-j2\pi},\ X(3)e^{-j3\pi}\}$$

$$= \{4,\ -1(1)(-j2),\ 0(1),\ j2(-1)\}$$

$$= \{4,\ j2,\ 0,\ -j2\}.$$

(b) DFT of $x(-n)$

Using Time reversal (flipping) property of DFT, We have

$$DFT\{x(-n)\} = X(-K) = X^*(K)$$

$$= \{4, -j2, 0, j2\} = \{4, j2, 0, -j2\}$$

(c) Using the conjugation property of DFT, We have

$$DFT\{x^*(n)\} = X^*(-K)$$

$$= \{4, -j2, 0, j2\}^* = \{4, j2, 0, -j2\}.$$

Since DFT $\{x^*(n)\} =$ DFT$\{x(n)\}$, We can say that $x(n)$ is real valued.

(d) Using property of convolution of product of two signals, We have

$$DFT\{x(n)\,x(n)\} = \frac{1}{N}\left[X(K) \circledast X(K)\right]$$

$$= \frac{1}{4}\left[(4, -j2, 0, j2) \circledast (4, -j2, 0, j2)\right]$$

By performing circular convolution between $(4, -j2, 0, j2) \circledast (4, -j2, 0, j2)$

$$\begin{bmatrix} 4 & j2 & 0 & -j2 \\ -j2 & 4 & j2 & 0 \\ 0 & -j2 & 4 & j2 \\ j2 & 0 & -j2 & 4 \end{bmatrix} \begin{bmatrix} 4 \\ -j2 \\ 0 \\ j2 \end{bmatrix} = \begin{bmatrix} 16+4+0+4 \\ -8j-8j+0+0 \\ 0-4+0-4 \\ 8j+0+0+8j \end{bmatrix} = \begin{bmatrix} 24 \\ -16j \\ -8 \\ 16j \end{bmatrix}$$

$$(4, -j2, 0, j2) \circledast (4, -j2, 0, j2) = (24, -16j, -8, 16j)$$

$$DFT\{x(n)\, x(n)\} = \frac{1}{N}[X(k) \circledast X(k)]$$

$$= \frac{1}{4}[24, -16j, -8, 16j] = [6, -4j, -2, 4j]$$

(e) DFT of $x(n) \circledast x(n)$

Using the circular convolution property of DFT, we have

$$DFT[x(n) \circledast x(n)] = [X(k)\, X(k)]$$

$$= \{(4, -j2, 0, j2)(4, -j2, 0, j2)\}$$

$$= \{16, -4, 0, -4\}.$$

(f) Signal energy

Using Parseval's theorem, we have

$$\text{Signal energy} = \frac{1}{4} \Sigma |X(k)|^2$$

$$= \frac{1}{4} \Sigma |\{4, -j2, 0, j2\}|^2 = \frac{1}{4} \Sigma |[16, -4, 0, -4]|$$

$$= \frac{1}{4}[16+4+0+4] = \frac{24}{4} = 6$$

If IDFT $\{X(K)\} = x(n) = \{1, 2, 1, 0\}$ using properties of DFT, find

(a) IDFT $\{X(K-1)\}$

(b) IDFT $\{X(K) \circledast X(K)\}$

(c) IDFT $\{X(K)X(K)\}$

(d) signal energy

Sol. (a) using circular frequency shift property

$$\text{IDFT}\{X(K-1)\} = x(n)e^{j2\pi n/4} = x(n)e^{jn\pi/2}$$

$$= \{x(0),\ x(1)e^{j\pi/2},\ x(2)e^{j\pi},\ x(3)e^{j3\pi/2}\}$$

$$= \{1,\ 2(0+j),\ 1(-1+j(0)),\ 0\} = \{1, 2j, -1, 0\}.$$

(b) IDFT $\{X(K) \circledast X(K)\}$.

using periodic convolution property, we have

$$\text{IDFT}\{X(K) \circledast X(K)\} = N x^2(n)$$

$$= 4[(1, 2, 1, 0) \times (1, 2, 1, 0)]$$

$$= 4[1, 4, 1, 0] = [4, 16, 4, 0]$$

(c) IDFT $\{X(K)X(K)\}$

using the convolution in time domain property, we have

$$\text{IDFT}\{X(K)X(K)\} = \{1, 2, 1, 0\} \circledast \{1, 2, 1, 0\}.$$

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+0+1+0 \\ 2+2+0+0 \\ 1+4+1+0 \\ 0+2+2+0 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 6 \\ 4 \end{bmatrix} = \{2, 4, 6, 4\}$$

(d) Signal energy $= \sum_{n=0}^{N-1} |x(n)|^2 = |x(0)|^2 + |x(1)|^2 + |x(2)|^2 + |x(3)|^2$

$$= 1 + 4 + 1 + 0$$

$$= \underline{6}$$

DISCRETE FOURIER SERIES

The Fourier series representation of a periodic discrete-time sequence is called Discrete Fourier series (DFS).

Consider a discrete-time signal $x(n)$, that is periodic with period 'N' defined by $x(n) = x(n+kN)$ for any integer value of 'k'.

The periodic function $x(n)$ can be synthesized as the sum of sine and cosine sequences (Trigonometric form of Fourier series) or equivalently as a linear combination of complex exponentials (Exponential form of Fourier series) whose frequencies are multiples of the fundamental frequency $\omega_0 = 2\pi/N$. This is done by constructing a periodic sequence for which each period is identical to the finite length sequence.

Exponential form of Discrete Fourier Series

The exponential form of the Fourier series for a periodic discrete-time signal is given by

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) e^{j(2\pi/N)nk} \quad \text{for all } n \qquad -①$$

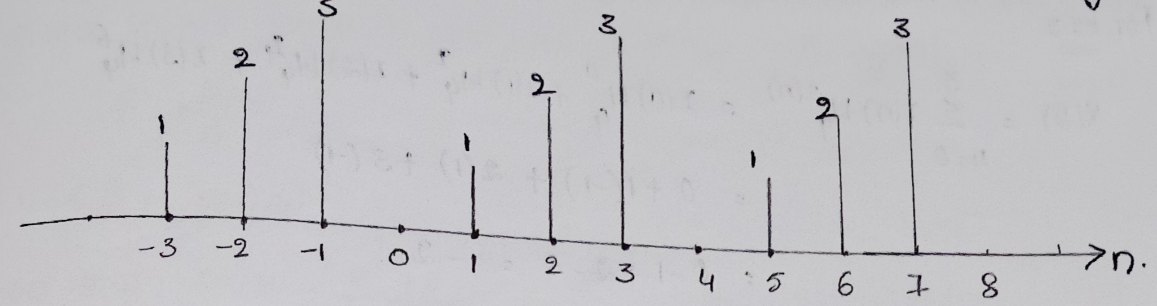
Where the co-efficients $X(k)$ are expressed as

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j(2\pi/N)nk} \quad \text{for all } k \qquad -②$$

The eq ① & ② are called DFS synthesis and analysis pair. Hence $X(k)$ and $x(n)$ are periodic sequences

The equivalent form for $X(k)$ is $\sum_{n=0}^{N-1} x(n) W_N^{nk}$ where $W_N = e^{-j(2\pi/N)}$ called the twiddle factor

Find the exponential form of the DFS representation of $x(n)$ shown in given fig



To determine the exponential form of DFS, We have

$$W_N^k = e^{-j(2\pi/N)k}$$

Given $\underline{N=4}$ → $W_4^0 = e^0 = 1$

$$W_4^1 = e^{-j2\pi/4} = e^{-j\pi/2} = \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} = -j$$

$$W_4^2 = e^{-j2\pi/4 \cdot 2} = e^{-j\pi} = \cos\pi - j\sin\pi = -1$$

$$W_4^3 = e^{-j2\pi/4 \cdot 3} = e^{-j3\pi/2} = \cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2} = j$$

$$W_4^4 = e^{-j2\pi/4 \cdot 4} = e^{-j2\pi} = \cos 2\pi - j\sin 2\pi = 1$$

The exponential form of DFS is given by

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j(2\pi/N)nk} = \frac{1}{N}\sum_{k=0}^{N-1} X(k)W_N^{-nk} \quad \text{for all } n$$

where $$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j(2\pi/N)nk} = \sum_{n=0}^{N-1} x(n)W_N^{nk} \quad \text{for all } n.$$

For $k=0$ $$X(0) = \sum_{n=0}^{3} x(n)W_4^0$$

$$= x(0) + x(1) + x(2) + x(3)$$

$$= 0 + 1 + 2 + 3 = \underline{6}$$

For $\underline{k=1}$ $$X(1) = \sum_{n=0}^{3} x(n)W_N^{x(n)} = \sum_{n=0}^{3} x(n)W_4^n$$

$$= x(0)W_4^0 + x(1)W_4^1 + x(2)W_4^2 + x(3)W_4^3$$

$$= 0(1) + 1(-j) + 2(-1) + 3(j) = 0 - j - 2 + 3j = -2 + 2j$$

For $k=2$

$$X(2) = \sum_{n=0}^{3} x(n) W_4^{2(n)} = x(0) W_4^{0} + x(1) W_4^{2} + x(2) W_4^{4} + x(3) W_4^{6}$$

$$= 0 + 1(-1) + 2(1) + 3(-1)$$

$$= 0 - 1 + 2 - 3 = -2$$

For $k=3$

$$X(3) = \sum_{n=0}^{3} x(n) W_4^{3n} = x(0) W_4^{0} + x(1) W_4^{(3)} + x(2) W_4^{6} + x(3) W_4^{9}$$

$$= 0 + 1(j) + 2(-1) + 3(-j) = -2 - j2$$

The complex exponential form of Fourier Series is

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_4^{-nk} = \frac{1}{4} \sum_{k=0}^{3} X(k) W_4^{-nk}.$$

$$= \frac{1}{4}\left[X(0) + X(1) W_4^{-n} + X(2) W_4^{-2n} + X(3) W_4^{-3n}\right]$$

$$= \frac{1}{4}\left[6 + (-2+j2) e^{j(2\pi/4)n} + (-2) e^{j(2\pi/4)2n} + (-2-j2) e^{j(2\pi/4)3n}\right]$$

$$= \frac{1}{4}\left[6 + (-2+j2) e^{j\pi/2 n} + (-2) e^{j\pi n} + (-2-j2) e^{j(3\pi/2)n}\right]$$

$$x(n) = \frac{1}{2}\left[3 + (-1+j) e^{(j\pi/2 n)} - e^{j\pi n} - (1+j) e^{j(3\pi/2)n}\right]$$

→ Determine DFS representation for the signal $x(n) = \cos\left(\frac{n\pi}{3}\right)$. Plot the Spectrum of $x(n)$

Sol. The given signal is $x(n) = \cos \frac{n\pi}{3}$

Cosine function can also be written as $x(n) = \cos(2\pi f n)$

Comparing the two sequences or equation then

$$2\pi f n = \frac{n\pi}{3}$$

$$f = \frac{1}{6}$$

Thus frequency is the ratio of two integers.

Hence given cosine wave is periodic. And for such signal,

$$f = \frac{k}{N} = \frac{1}{6}$$

Hence $N = 6$, i.e period

We know that $\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$ — ①

$$\cos\frac{n\pi}{3} = \frac{1}{2}\left[e^{jn\pi/3} + e^{-j\frac{n\pi}{3}}\right] \quad -②$$

We know that

$$x(n) = \frac{1}{N}\sum_{k=\langle N\rangle} X(k)\, e^{j2\pi nk/N} \quad -③$$

For $N=6$ then

$$x(n) = \sum_{k=-2}^{3} X(k)\, e^{j2\pi kn/6} = \sum_{k=-2}^{3} X(k) e^{j(\frac{\pi}{3})kn}$$

$$= X(-2)\, e^{-j(\frac{\pi}{3})2n} + X(-1)\, e^{-j(\frac{\pi}{3})n} + X(0)e^{0} + X(1)e^{j(\frac{\pi}{3})n}$$

$$+ X(2)e^{j(\frac{\pi}{3})2n} + X(3)e^{j(\frac{\pi}{3})3n} \quad -④$$

By comparing the eq ② & ④, We get

$X(-2) = 0$; $X(-1) = \frac{1}{2}$; $X(0) = 0$; $X(1) = \frac{1}{2}$; $X(2) = 0$; $X(3) = 0$

other values of $X(k)$ can be calculated easily since $X(k)$ repeats after the period N.

$X(-2) = X(-2+6) = X(4) = 0$

$X(-1) = X(-1+6) = X(5) = \frac{1}{2}$

$X(0) = X(0+6) = X(6) = 0$

$X(1) = X(1+6) = X(7) = \frac{1}{2}$

$X(2) = X(2+6) = X(8) = 0$

$X(3) = X(3+6) = X(9) = 0$

Similarily $C(-2) = C(-2-6) = C(-8) = 0$

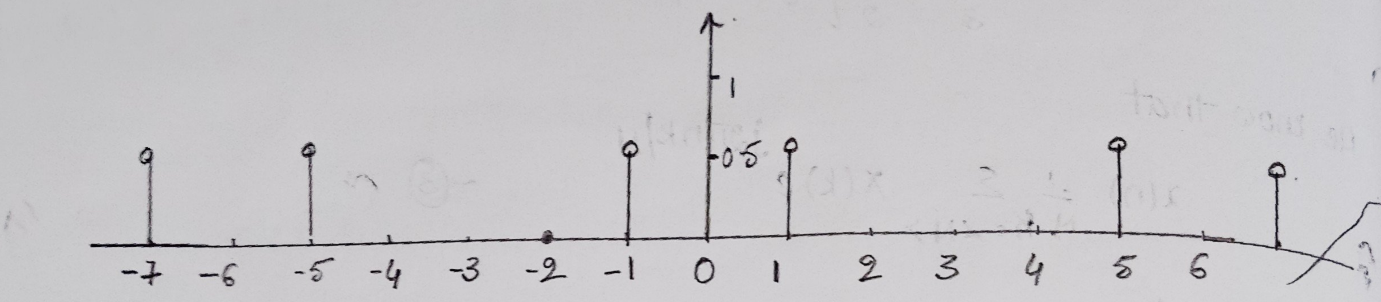
$C(-1) = C(-1-6) = C(-7) = \frac{1}{2}$

$C(0) = C(0-6) = C(-6) = 0$

$C(1) = C(1-6) = C(-5) = \frac{1}{2}$

$C(2) = C(2-6) = C(-4) = 0$

$C(3) = C(3-6) = C(-3) = 0$ and so on



Spectrum of $\cos\left(\frac{n\pi}{3}\right)$

→ Distinguish between DFT and DTFT

DFT	DTFT
1. obtained by performing sampling operation in both the time and frequency domain.	1. Sampling is performed only in time domain.
2. Discrete frequency spectrum	2. Continous function of ω

→ Distinguish between Fourier series and Fourier transform

Fourier series	Fourier Transform.
1. Gives the harmonic content of a periodic time function	1. Gives the frequency information for an aperiodic signal.
2. Discrete frequency spectrum A discrete distribution is one in which	2. Continous frequency spectrum.

Methods to evaluate circular convolution of two sequences

The methods to find the circular convolution of two sequences are

1. Concentric circle Method
2. Matrix multiplication method.

Concentric Circle Method

Given two sequences $x_1(n)$ and $x_2(n)$, the circular convolution of these two sequences $x_3(n) = x_1(n) \textcircled{N} x_2(n)$ can be found by using the following steps.

1. Graph N samples of $x_1(n)$ as equally spaced points around an outer circle in counterclockwise direction
2. start at the same point as $x_1(n)$ graph 'N' samples of $x_2(n)$ as equally spaced points around an inner circle in clockwise direction
3. Multiply corresponding samples on the two circles and sum the products to produce output.
4. Rotate the inner circle one sample at a time in counter clockwise direction and go to step 3 to obtain the next value of output.
5. Repeat step No. 4 until the inner circle first sample lines up with the first sample of the exterior circle once again.

Matrix Multiplication Method

In this method, the circular convolution of two sequences $x_1(n)$ and $x_2(n)$ can be obtained by representing the sequences in matrix form as shown below:

$$
\begin{bmatrix}
x_2(0) & x_2(N-1) & x_2(N-2) & \cdots & x_2(2) & x_2(1) \\
x_2(1) & x_2(0) & x_2(N-1) & \cdots & x_2(3) & x_2(2) \\
x_2(2) & x_2(1) & x_2(0) & \cdots & x_2(0) & x_2(3) \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
x_2(N-2) & x_2(N-3) & x_2(N-4) & \cdots & x_2(0) & x_2(N-1) \\
x_2(N-1) & x_2(N-2) & x_2(N-3) & \cdots & x_2(1) & x_2(0)
\end{bmatrix}
\begin{bmatrix}
x_1(0) \\ x_1(1) \\ x_2(1) \\ \vdots \\ x_2(N-2) \\ x_1(N-1)
\end{bmatrix}
=
\begin{bmatrix}
x_3(0) \\ x_3(1) \\ x_3(2) \\ \vdots \\ x_3(N-2) \\ x_3(N-1)
\end{bmatrix}
$$

The sequence $x_2(n)$ is repeated via circular shift of samples of samples and represented in N X N matrix form. The sequence $x_1(n)$ is represented as column matrix. The multiplication of these two matrices gives the sequences $x_3(n)$

→ Find the circular convolution of two finite duration sequences

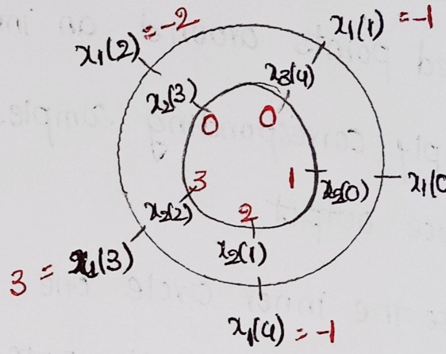
$x_1(n) = \{1, -1, -2, 3, -1\}$; $x_2(n) = \{1, 2, 3\}$.

Sol. To find circular convolution, both the sequences must be of same length. Therefore we append two zeros to the sequence $x_2(n)$ and use of concentric circle method to find circular convolution.

We have

$x_1(n) = \{1, -1, -2, 3, -1\}$

$x_2(n) = \{1, 2, 3, 0, 0\}$



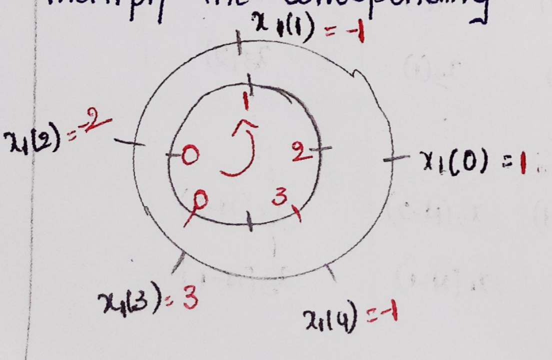
Graph all the points of $x_1(n)$ on the outer circle in the counter clockwise direction. starting at same point as $x_1(n)$ graph all points of $x_2(n)$ on the inner circle in clockwise direction.

Multiply corresponding samples on the circle and add to obtain

$y(0) = 1(1) + 0(-1) + 0(-2) + 3(3) + 2(-1)$

$= 1 + 0 + 0 + 9 - 2 = \underline{\underline{8}}$

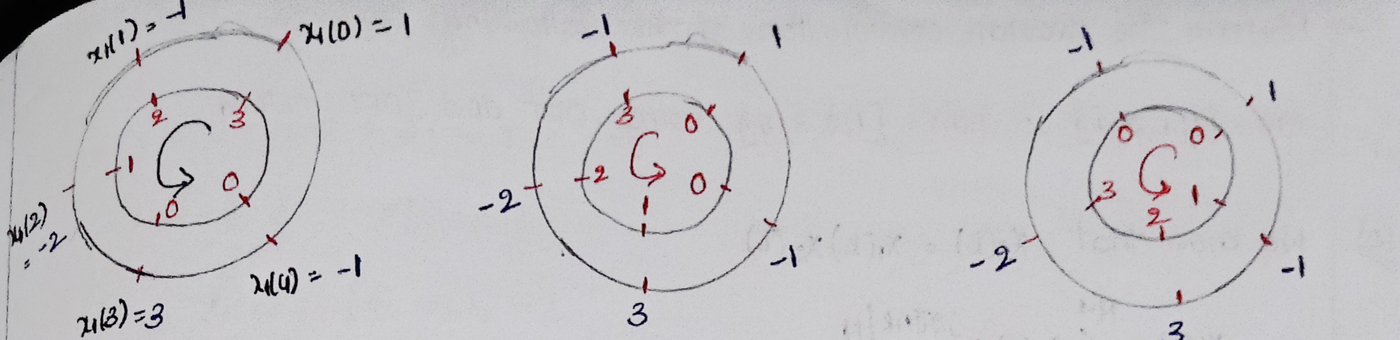
Rotate the inner circle in counterclose wise direction by one sample, multiply the corresponding samples to obtain $y_1(1)$



$y(1) = 2(1) - 1(1) + 0(-2) + 0(3) + 3(-1)$

$= 2 - 1 + 0 + 0 - 3$

$= 2 - 4 = \underline{\underline{-2}}$



$y(2) = 1(3) - 1(2) - 2(1) + 3(0) - 1(0)$

$= 3 - 2 - 2 + 0 + 0$

$= -1$

$y(3) = 1(0) - 1(3) - 2(2) + 3(1) + 0(-1)$

$= 0 - 3 - 4 + 3 + 0$

$= -4$

$y(5) = 1(0) - 1(0) + 3(-2) + 2(3) + 1(-1)$

$= 0 - 0 - 6 + 6 - 1 = -1.$

$\therefore y(n) = \{8, -2, -1, -4, -1\}$.

Matrix method

Given $x_1(n) = \{1, -1, -2, 3, -1\}$

$x_2(n) = \{1, 2, 3\}$

By adding two zeros to the sequence $x_2(n)$, we bring the length of the sequence $x_2(n)$ to 5

Now $x_2(n) = \{1, 2, 3, 0, 0\}$.

The matrix form can be written by substituting $N = 5$

$$\begin{bmatrix} x_2(0) & x_2(4) & x_2(3) & x_2(2) & x_2(1) \\ x_2(1) & x_2(0) & x_2(4) & x_2(3) & x_2(2) \\ x_2(2) & x_2(1) & x_2(0) & x_2(4) & x_2(3) \\ x_2(3) & x_2(2) & x_2(1) & x_2(0) & x_2(4) \\ x_2(4) & x_2(3) & x_2(2) & x_2(1) & x_2(0) \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_1(1) \\ x_1(2) \\ x_1(3) \\ x_1(4) \end{bmatrix} = \begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \\ y(4) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 3 & 2 \\ 2 & 1 & 0 & 0 & 3 \\ 3 & 2 & 1 & 0 & 0 \\ 0 & 3 & 2 & 1 & 0 \\ 0 & 0 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 1+0+0+9+(-2) \\ 2-1+0+0-3 \\ 3-2-2+0+0 \\ 0-3-4+3+0 \\ 0+0-6+6-1 \end{bmatrix} = \begin{bmatrix} 8 \\ -2 \\ -1 \\ -4 \\ -1 \end{bmatrix}$$

→ Perform the circular convolution of the following sequences

Prev paper problem

$x(n) = \{1,1,2,1\}$; $h(n) = \{1,2,3,4\}$ using DFT and IDFT method.

Sol. We know that $X_3(k) = X_1(k) X_2(k)$

$$X_1(k) = \sum_{n=0}^{N-1} x_1(n) e^{-j2\pi nk/N}$$

For $k=0$

$$X_1(0) = \sum_{n=0}^{3} x_1(n) e^{0} = x_1(0) + x_1(1) + x_2(2) + x_1(3)$$

$$= 1+1+2+1 = 5$$

For $k=1$

$$X_1(1) = \sum_{n=0}^{3} x_1(n) e^{-j2\pi n/4} = \sum_{n=0}^{3} x_1(n) e^{-jn\pi/2}$$

$$= x_1(0) + x_1(1) e^{-j\pi/2} + x_1(2) e^{-j\pi} + x_1(3) e^{-j3\pi/2}$$

$$= 1 + e^{-\pi/2} + 2e^{-j\pi} + e^{-j3\pi/2}$$

$$= 1 + \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} + 2(\cos\pi - j\sin\pi) + \cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2}$$

$$= 1 + 0 - j + 2(-1) - 2j(0) + 0 - j(-1)$$

$$= 1 - j - 2 - 2j(0) + j = \underline{-1}$$

For $k=2$

$$X_1(2) = \sum_{n=0}^{3} x_1(n) e^{-j2\pi n 2/4} = \sum_{n=0}^{3} x_1(n) e^{-jn\pi}$$

$$= x_1(0) + x_1(1) e^{-j\pi} + x_1(2) e^{-j2\pi} + x_1(3) e^{-j3\pi}$$

$$= 1 + \cos\pi - j\sin\pi + 2(\cos 2\pi - j\sin 2\pi) + \cos 3\pi - j\sin 3\pi$$

$$= 1 + (-1) - j(0) + 2(1 - j(0)) + (-1) + (0)$$

$$= 1 - 1 - 0 + 2 - 1 = \underline{1}$$

For $k=3$

$$X_1(3) = \sum_{n=0}^{3} x_1(n) e^{-j2\pi \cdot 3n/4} = \sum_{n=0}^{3} x_1(n) e^{-j3\pi n/2}$$

$$= x_1(0) + x_1(1) e^{-j3\pi/2} + x_1(2) e^{-j3\pi} + x_1(3) e^{-j9\pi/2}$$

$$X_1(3) = 1+\cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2} + 2[\cos 3\pi - j\sin 3\pi] + [\cos\frac{9\pi}{2} - j\sin\frac{9\pi}{2}]$$

$$= 1+0-j(-1)+2(-1)+0-j(1)$$

$$= 1+j-2-j = \underline{\underline{-1}}$$

$$\therefore X_1(K) = \{5, -1, 1, -1\}.$$

Given $x_2(n) = \{1,2,3,4\}$

$$X_2(0) = \sum_{n=0}^{N-1} x_2(n) e^{-j2\pi nk/N} = \sum_{n=0}^{3} x_2(n) e^{0} = x_2(0)+x_2(1)+x_2(2)+x_2(3)$$

$$= 1+2+3+4 = 10$$

$$X_2(1) = \sum_{n=0}^{N-1} x_2(n) e^{-j2\pi n/4} = \sum_{n=0}^{N-1} x_2(n) e^{-jn\pi/2}$$

$$= x_2(0) + x_2(1)e^{-j\pi/2} + x_2(2)e^{-j\pi} + x_2(3)e^{-j3\pi/2}$$

$$= 1+2e^{-j\pi/2} + 3e^{-j\pi} + 4e^{-j3\pi/2}$$

$$= 1+2[\cos\frac{\pi}{2} - j\sin\frac{\pi}{2}] + 3[\cos\pi - j\sin\pi] + 4[\cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2}]$$

$$= 1+2[0-j] + 3[-1-j(0)] + 4[0-j(-1)]$$

$$= 1-2j-3+4j = \underline{\underline{-2+2j}}$$

$$X_2(2) = \sum_{n=0}^{N-1} x_2(n) e^{-j2\pi 2n/4} = \sum_{n=0}^{3} x_2(n) e^{-jn\pi}$$

$$= x_2(0) + x_2(1)e^{-j\pi} + x_2(2)e^{-j2\pi} + x_2(3)e^{-j3\pi}$$

$$= 1+2[\cos\pi - j\sin\pi] + 3[\cos 2\pi - j\sin 2\pi] + 4[\cos 3\pi - j\sin 3\pi]$$

$$= 1+2[(-1)-j(0)] + 3[1-j(0)] + 4[(-1)-j(0)]$$

$$= 1-2+3-4 = \underline{\underline{-2}}$$

$$X_2(3) = \sum_{n=0}^{N-1} x_2(n) e^{-j2\pi 3n/4} = \sum_{n=0}^{N-1} x_2(n) e^{-j3n\pi/2}$$

$$= x_2(0) + x_2(1)e^{-j3\pi/2} + x_2(2)e^{-j3\pi} + x_2(3)e^{-j9\pi/2}$$

$$= 1+2[\cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2}] + 3[\cos 3\pi - j\sin 3\pi] + 4[\cos\frac{9\pi}{2} - j\sin\frac{9\pi}{2}]$$

$x_2(3) = [1 + 2(0) - 2j(-1)] + 3(-1) - 3j(0) + 4.[0 - j(1)]$

$= 1 + 2j - 3 - 4j = -2 - 2j$

$X_2(k) = \{10, -2+j2, -2, -2-j2\}.$

$X_3(k) = X_1(k) . X_2(k)$

$= \{(5, -1, 1, -1) \cdot (10, -2+j2, -2, -2-j2)\}$

$X_3(k) : \{50, 2-j2, -2, 2+j2\}.$

$x_3(n) = \frac{1}{N} \sum_{k=0}^{N-1} X_3(k) e^{j2\pi nk/N}$ for $n = 0, 1 \ldots N-1$

n=0

$x_3(0) = \frac{1}{4} \sum_{k=0}^{3} X_3(k) e^{0} = \frac{1}{4}[50 + 2 - \cancel{j2} - \cancel{2} + \cancel{2} + \cancel{j2}] = \frac{1}{4}[52] = 13$

n=1

$x_3(1) = \frac{1}{4} \sum_{k=0}^{3} X_3(k) e^{j2\pi k/4} = \frac{1}{4}\left[\sum_{k=0}^{3} X_3(k) e^{jk\pi/2}\right]$

$= \frac{1}{4}[X_3(0) + X_3(1)e^{j\pi/2} + X_3(2)e^{j\pi} + X_3(3)e^{j3\pi/2}]$

$= \frac{1}{4}[50 + (2-j2)(\cos\frac{\pi}{2} + j\sin\frac{\pi}{2}) - 2(\cos\pi + j\sin\pi) + (2+j2)(\cos\frac{3\pi}{2} + j\sin\frac{3\pi}{2})]$

$= \frac{1}{4}[50 + (2-j2)(0+j) - 2(-1 + j(0)) + (2+j2)(0 + j(-1))$

$= \frac{1}{4}[50 + \cancel{2}j + 2 + \cancel{2} + \cancel{2} - \cancel{j2}] = \frac{56}{4} = 14$

n=2

$X_3(2) = \frac{1}{4} \sum_{k=0}^{3} X_3(k) e^{j2\pi 2k/4} = \frac{1}{4} \sum_{k=0}^{3} X_3(k) e^{j2\pi k/2 \; j\pi k}$

$= \frac{1}{4}[X_3(0) + X_3(1)e^{j\pi/2} + X_3(2)e^{j2\pi} + X_3(3)e^{j3\pi/2}]$

$= \frac{1}{4}[50 + (2-j2)(\cos\frac{\pi}{2} + j\sin\frac{\pi}{2}) - 2(\cos 2\pi + j\sin 2\pi) + (-2-j2)(\cos\frac{9\pi}{2} + j\sin\frac{9\pi}{2})]$ (3π, 3π)

$= \frac{1}{4}[50 + \overset{(-1+0)}{(2+j2)} \overset{(2-j2)}{(0+j)} - 2(+1 + j(0)) + (+2+j2)(-0+j)]$

$$x_3(2) = \frac{1}{4}\left[\cancel{50+2j+2+2-2j+2}\right]\left[50-2+\cancel{j2}-2-2-\cancel{j2}\right]$$

$$= \frac{1}{4}\left[50-6\right] = \frac{44}{4} = 11.$$

$$x_3(3) = \frac{1}{4}\left[\sum_{k=0}^{3} X_3(k)\, e^{j3\pi k/2}\right]$$

$$= \frac{1}{4}\left[50 + (2-j2)(-j) + (-2)(-1) + (2+j2)(j)\right] = 12$$

$$\therefore y_3(n) = \{13, 14, 11, 12\}$$

Filtering Long Duration Sequence

Suppose an input sequence $x(n)$ of long duration is to be processed with a system having impulse response of finite duration by convolving the two sequences. Because of the length of the input sequence, it would be not practical to store it all before performing linear convolution.

Therefore, the input sequence must be divided into blocks. The successive blocks are processed separately one at a time and the results are combined later to yield the desired output sequence which is identical to the sequence obtained by linear convolution.

Two methods that are commonly used for filtering the sectioned data and combining the results are the overlap-save method and the overlap-add method.

Overlap - Save Method

Let the length of an input sequence be L_s and the length of an impulse response is M. In this method, the input sequence is divided into blocks of data size $N = L+M-1$.

Each block consists of last $(M-1)$ data points of previous block followed by 'L' new data points to form a data sequence of length $N = L+M-1$.

For first block of data the first $(M-1)$ points are set to zero. Thus the block of data sequence are

$$x_1(n) = \{\underbrace{0,0,0,\ldots 0}_{(M-1)\ \text{zeros}}, x(0), x(1), \ldots x(L-1)\}$$

$$x_2(n) = \{\underbrace{x(L-M+1), \ldots x(L-1)}_{\text{Last } (M-1) \text{ data points from } x_1(n)}, x(L), \ldots x(2L-1)\}$$

$$x_3(n) = \{\underbrace{x(2L-M+1), \ldots x(2L-1)}_{\text{Last } (M-1) \text{ data points from } x_2(n)}, x(2L) \ldots x(3L-1)\}$$ and so on

Now, the impulse response of the FIR filter is increased in length by appending $(L-1)$ zeros and an N-point circular convolution of $x_i(n)$ with $h(n)$ is computed. i.e

$$y_i(n) = x_i(n) \; N \; h(n)$$

→ **Overlap - Add Method**

Let the length of the sequence be L_s and the length of the impulse response is M. The sequence is divided into blocks of data size having length 'L' and M-1 zeros are appended to it to make the data size of L+M-1.

Thus, the data blocks may be represented as

$$x_1(n) = \{x(0), x(1), \ldots x(L-1), \underbrace{0,0 \ldots}_{(M-1)\ \text{zeros appended}}\}$$

$$x_2(n) = \{x(L), x(L+1), \ldots x(2L-1), \underbrace{0,0 \ldots}\}$$

$$x_3(n) = \{x(2L), x(2L+1), \ldots x(3L-1), \underbrace{0,0 \ldots}_{(M-1)\ \text{zeros appended}}\}$$

Now, $(L-1)$ zeros are added to the impulse response $h(n)$ and N-point circular convolution is performed. Since each data block is terminated with (M-1) zeros, the last M-1 points from each output block must be overlapped and added to the first (M-1) points of the succeeding block, Hence this method is called overlap-add method.

Let the output block are of the form

$y_1(n) = \{ y_1(0), y_1(1), \dots y_1(L-1), y_1(L), \dots y_1(N-1) \}$

$y_2(n) = \{ y_2(0), y_2(1), \dots y_2(L-1), y_2(L) \dots y_2(N-1) \}$

$y_3(n) = \{ y_3(0), y_3(1), \dots y_3(L-1), y_3(L) \dots y_3(N-1) \}$

The output sequence is

$y(n) = \{ y_1(0), y_1(1), \dots y_1(L-1), y_1(L)+y_2(0), \dots y_1(N-1)+y_2(M-2),$

$y_2(M), \dots y_2(L)+y_3(0), y_2(L+1)+y_3(1)+\dots y_3(N-1) \}$

→ Find the output y(n) of a filter whose impulse response is $h(n) = \{1,1,1\}$ and input signal $x(n) = \{3,-1,0,1,3,2,0,1,2,1\}$ using (i) overlap-save method (ii) overlap-add method.

Sol (i) Overlap-save Method

Given $x(n) = L_s = \{3,-1,0,1,3,2,0,1,2,1\}$.

$h(n) = M = \{1,1,1\}$.

$N = 2^M = 2^3 = 8$ = Total length of each block.

$x_1(n) = \{ \underbrace{0, 0}_{(M-1) \text{ zeros}}, 3, -1, 0, 1, 3, 2 \}$

$x_2(n) = \{ \underbrace{3, 2}_{\text{Two data points from previous block}}, 0, 1, 2, 1, 0, 0 \}$

$h(n) = \{1,1,1,0,0,0,0,0\}$

$y_1(n) = x_1(n) \, (N) \, h(n)$

$y_2(n) = x_2(n) \, (N) \, h(n)$

$y_1(n) = x_1(n) \; (N) \; h(n)$

$= \{0, 0, 3, -1, 0, 1, 3, 2\} \; (N) \; \{1, 1, 1, 0, 0, 0, 0, 0\}.$

$$\begin{bmatrix} 0 & 2 & 3 & 1 & 0 & -1 & 3 & 0 \\ 0 & 0 & 2 & 3 & 1 & 0 & -1 & 3 \\ 3 & 0 & 0 & 2 & 3 & 1 & 0 & -1 \\ -1 & 3 & 0 & 0 & 2 & 3 & 1 & 0 \\ 0 & -1 & 3 & 0 & 0 & 2 & 3 & 3 \\ 1 & 0 & -1 & 3 & 0 & 0 & 2 & 3 \\ 3 & 1 & 0 & -1 & 3 & 0 & 0 & 2 \\ 2 & 3 & 1 & 0 & -1 & 3 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2+3 \\ 2 \\ 3 \\ -1+3 \\ -1+3 \\ 1-1 \\ 3+1 \\ 2+3+1 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 3 \\ 2 \\ 2 \\ 0 \\ 4 \\ 6 \end{bmatrix}$$

$\therefore y_1(n) = \{5, 2, 3, 2, 2, 0, 4, 6\}.$

$y_2(n) = x_2(n) \; (N) \; h(n)$

$= \{3, 2, 0, 1, 2, 1, 0, 0\} \; (N) \; \{1, 1, 1, 0, 0, 0, 0, 0\}.$

$$\begin{bmatrix} 3 & 0 & 0 & 1 & 2 & 1 & 0 & 2 \\ 2 & 3 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 2 & 3 & 0 & 0 & 1 & 2 & 1 \\ 1 & 0 & 2 & 3 & 0 & 0 & 1 & 2 \\ 2 & 1 & 0 & 2 & 3 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 & 2 & 3 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 2 & 3 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 2+3 \\ 0+2+3 \\ 1+2 \\ 2+1 \\ 1+2+1 \\ 1+2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 5 \\ 3 \\ 3 \\ 4 \\ 3 \\ 1 \end{bmatrix}$$

$\therefore y_2(n) = \{3, 5, 5, 3, 3, 4, 3, 1\}$

$y(n) = \{\underbrace{5, 2}_{\text{Discard}}, 3, 2, 2, 0, 4, 6\}$

$\{\underbrace{3, 5}_{\text{Discard}}, 5, 3, 3, 4, 3, 1\}$

$\therefore y(n) = \{3, 2, 2, 0, 4, 6, 5, 3, 3, 4, 3, 1\}$

(ii) Over-lap add Method

Given $x(n) = \{3, -1, 0, 1, 3, 2, 0, 1, 2, 1\} : L_s = 10$

$h(n) = \{1, 1, 1\} \therefore M = 3$

$M = 3 \Rightarrow N = 2^3 = 8$

$N = L + M - 1$

$8 = L + 3 - 1 \Rightarrow L = 6$

$\therefore y_1(n) = x_1(n) \, (N) \, h(n)$

$y_2(n) = x_2(n) \, (N) \, h(n)$

$x_1(n) = \{3, -1, 0, 1, 3, 2, 0, 0\}$

$x_2(n) = \{0, 1, 2, 1, 0, 0, 0, 0\}$

$h(n) = \{1, 1, 1, 0, 0, 0, 0, 0\}$.

$y_1(n) = \{3, -1, 0, 1, 3, 2, 0, 0\} \, (N) \, \{1, 1, 1, 0, 0, 0, 0, 0\}$.

$$\begin{bmatrix} 3 & 0 & 0 & 2 & 3 & 1 & 0 & -1 \\ -1 & 3 & 0 & 0 & 2 & 3 & 1 & 0 \\ 0 & -1 & 3 & 0 & 0 & 2 & 3 & 1 \\ 1 & 0 & -1 & 3 & 0 & 0 & 2 & 3 \\ 3 & 1 & 0 & -1 & 3 & 0 & 0 & 2 \\ 2 & 3 & 1 & 0 & -1 & 3 & 3 & 0 \\ 0 & 2 & 3 & 1 & 0 & -1 & 3 & 0 \\ 0 & 0 & 2 & 3 & 1 & 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -1+3 \\ -1+3 \\ 1-1 \\ 3+1 \\ 2+3+1 \\ 2+3 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 2 \\ 0 \\ 4 \\ 6 \\ 5 \\ 2 \end{bmatrix}$$

$y_1(n) = \{3, 2, 2, 0, 4, 6, 5, 2\}$.

$y_2(n) = x_2(n) \, (N) \, h(n)$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2+1 \\ 1+2+1 \\ 0+1+2 \\ 0+0+1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 4 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$y_2(n) = \{0, 1, 3, 4, 3, 1, 0, 0\}$

In over-lap add Method

$y_1(n) \rightarrow$ 3 2 2 0 4 6 5 2

$y_2(n) \rightarrow$ 0 1 3 4 3 1 0 0

3 2 2 0 4 6 5 3 3 4 3 1 0 0

$\therefore y(n) = \{3, 2, 2, 0, 4, 6, 5, 3, 3, 4, 3, 1\}$

→ Fast Fourier Transform

The DFT of a sequence can be evaluated using the formula

$$X(K) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi nK/N} \qquad 0 \le K \le N-1$$

Substituting $W_N = e^{-j2\pi/N}$, We have

$$X(K) = \sum_{n=0}^{N-1} x(n) W_N^{nk} \qquad 0 \le K \le N-1$$

$$= \sum_{n=0}^{N-1} \{Re[x(n)] + j Img[x(n)]\} \{Re[W_N^{nk}] + j Im[W_N^{nk}]\}$$

$$= \sum_{n=0}^{N-1} Re[x(n)] Re[W_N^{nk}] - \sum_{n=0}^{N-1} Img[x(n)][W_N^{nk}] + j \left\{ \sum_{n=0}^{N-1} Im[x(n)] Re[W_N^{nk}] \right.$$

$$\left. + \sum_{n=0}^{N-1} Re[x(n)] Im[W_N^{nk}] \right.$$

We can see that to evaluate one value of $X(K)$, the no. of complex multiplications required is N. Therefore to evaluate all N value of $X(K)$, the no. of complex multiplications required is N^2.

In the same way, to evaluate one value of $X(K)$ the no. of complex additions required is $(N-1)$. To evaluate all the values of $X(K)$, the total no. of complex additions required is $N(N-1)$

Each of four sums of 'N' terms requires 'N-1' real two-input additions and to combine the sum to get the real part and imaginary part requires two more.

Therefore to evaluate $X(k)$ for each 'k' requires $4(N-1)+2$ real additions. For all values of 'k' a total no. of real additions $N(4N-2)$ required for direct evaluation of the DFT.

The FFT algorithm requires $\frac{N}{2}\log_2 N$ multiplications and $N\log_2 N$ additions respectively

<u>summary of steps of radix-2 DIT-FFT algorithm</u>

1. The no. of input samples $N=2^M$, where M is an integer
2. The input sequence is shuffled through bit-reversal
3. The no. of stages in the flowgraph is given by $M=\log_2 N$
4. Each stage consists of $\frac{N}{2}$ butterflies.
5. Inputs/outputs for each butterfly are separated by 2^{m-1} samples, where 'm' represents the stage index i.e for first stage $m=1$ and for second stage $m=2$ so on.
6. The no. of complex multiplications is given by $\frac{N}{2}\log_2 N$
7. The no. of complex additions is given by $N\log_2 N$
8. The twiddle factor exponents are a function of the stage index 'm' and is given by

$$k=\frac{Nt}{2^m} \quad t=0,1,2\ldots 2^{m-1}-1$$

9. The no. of sets or sections of butterflies in each stage is given by the formula 2^{M-m}
10. The exponent repeat factor (ERF), which is the no. of times the exponent sequence associated with 'm' is repeated is given by 2^{M-m}

→ Find the DFT of a sequence $x(n) = \{1,2,3,4,4,3,2,1\}$ using DIT algorithm

Sol. Given $N = 8 = 2^M$

$\therefore M = 3$ stages.

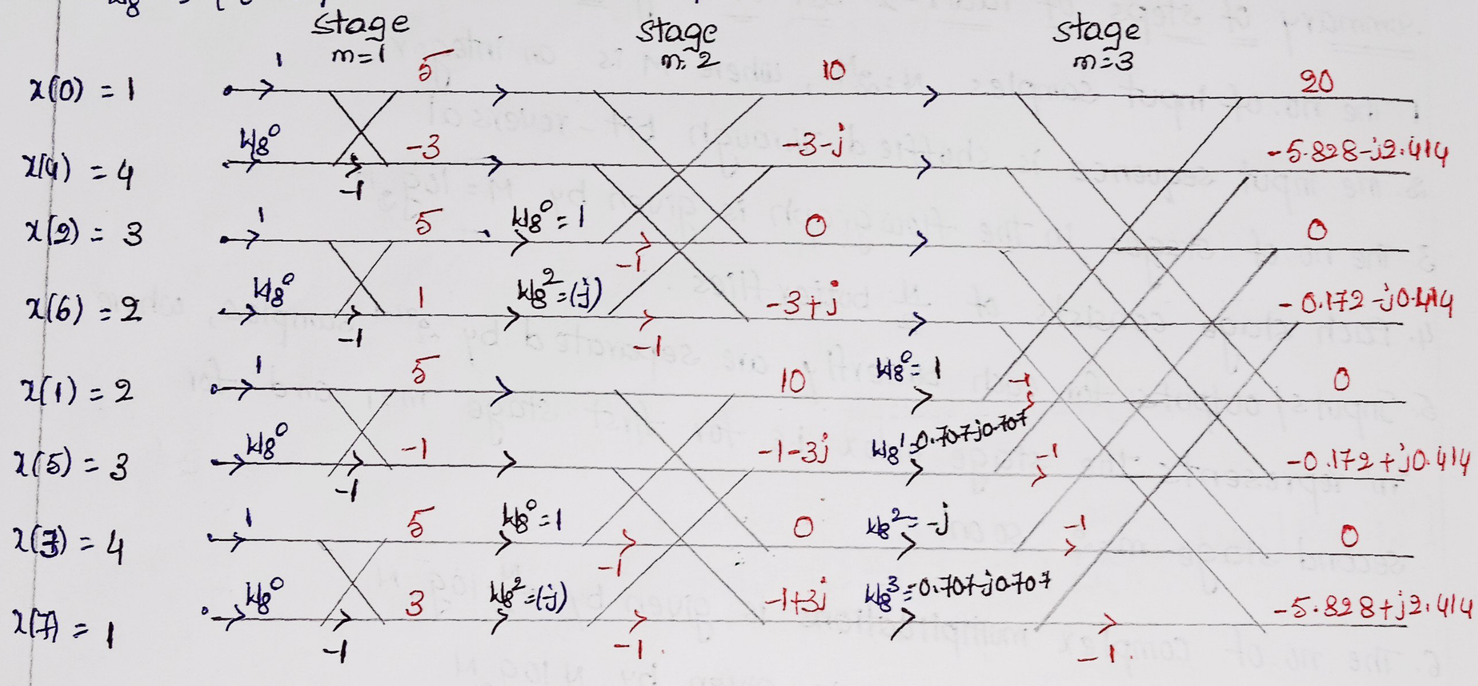
The twiddle factors associated with the flow graph are

$W_8^0 = 1$

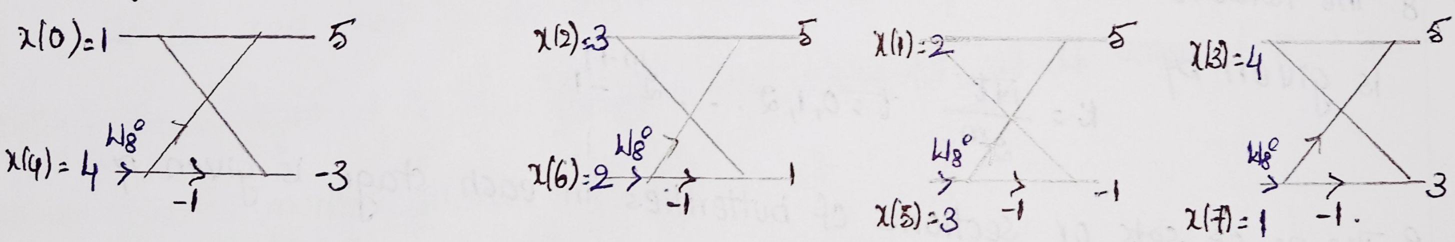
$W_8^1 = (e^{-j2\pi/8})^1 = e^{-j\pi/4} = \cos\frac{\pi}{4} - j\sin\frac{\pi}{4} = 0.707 - j0.707$

$W_8^2 = (e^{-j2\pi/8})^2 = e^{-j\pi/2} = \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} = 0 - j = -j$

$W_8^3 = (e^{-j\frac{2\pi}{8}})^3 = e^{-j3\pi/4} = \cos\frac{3\pi}{4} - j\sin\frac{3\pi}{4} = -0.707 - j0.707.$



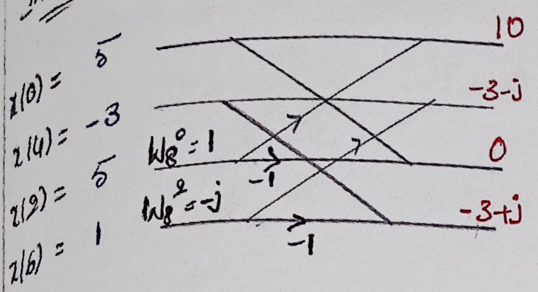
In stage 1

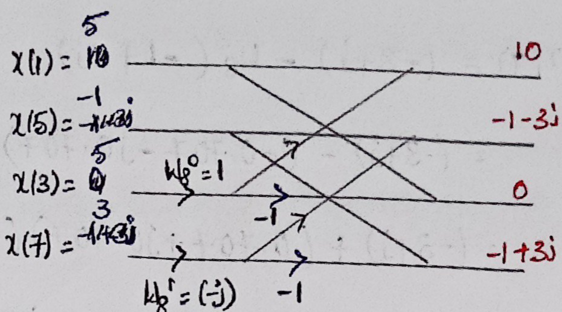


$1+4 = 5$ $\quad$ $3+2 = 5$ $\quad$ $2+3 = 5$ $\quad$ $4+1 = 5$

$1-4 = -3.$ $\quad$ $3-2 = 1$ $\quad$ $2-3 = -1$ $\quad$ $4-1 = 3$

In stage 2:





$x(0) \rightarrow 5+5 = 10$

$x(4) \rightarrow -3-j$

$x(2) \rightarrow 0$

$x(6) \rightarrow -3-1(-j) = -3+j$

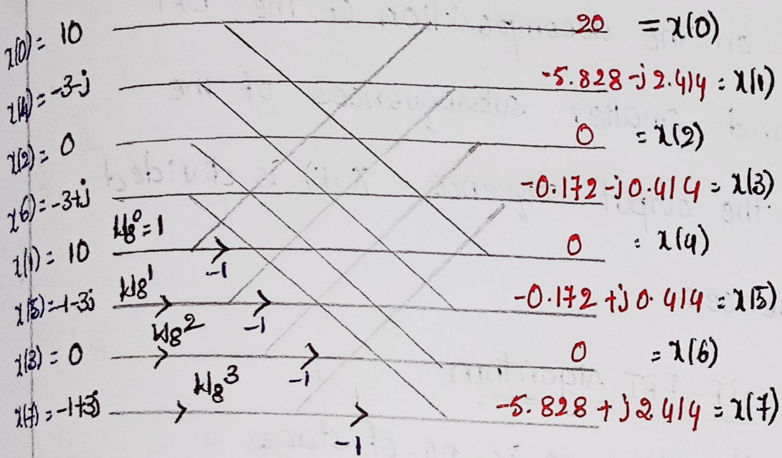
$x(1) \Rightarrow 10$

$x(5)' \Rightarrow -1-3j$

$x(3) \Rightarrow -5+5 = 0$

$x(7) \Rightarrow -1+3j$

In stage 3:



$x(0) = 10+10 = 20$

$x(4) = -3-j + W_8^1(-1-3j)$

$= -3-j+(0.707-j0.707)(-1-3j)$

$= -3-j - 0.707 - j2.121 + j0.707 - 2.121$

$= -3-0.707-2.121 - j(1+2.121-0.707)$

$= -5.828 - j2.414$

$x(2) = 0+0 = 0$

$x(6) = (-3+j) + W_8^3(-1+3j)$

$= (-3+j) + (-0.707-j0.707)(-1+3j)$

$= (-3+j) - (0.707+j0.707)(-1+3j)$

$= -3+j - [-0.707 + j2.121 - j0.707 - 2.121]$

$= -3+j+0.707-j2.121+j0.707+2.121$

$= -3+0.707+2.121+j(1-2.121+0.707)$

$= -0.172 - j0.414$

$x(1) = 10-10 = 0$

$x(5) = (-3-j) - W_8^1(-1-3j)$

$= (-3-j) + (0.707-j0.707)(1+3j)$

$= -3-j+0.707-j0.707 + j2.121 + 2.121$

$= -3+0.707+2.121 - j(1+0.707 - 2.121)$

$= -0.172+j0.414$

$x(3) = 0+0 = 0$.

$x(7) = (-3+j) - W_8^3(-1+3j)$

$= (-3+j) - (-0.707 - j0.707)(-1+3j)$

$= (-3+j) + (0.707 + j0.707)(-1+3j)$

$= -3+j - 0.707 - j0.707 + j2.121 - 2.121$

$= -3 - 0.707 - 2.121 + j(1 - 0.707 + 2.121)$

$= -5.828 + j2.414$

$\therefore X(k) = \{20, -5.828 - j2.414, 0, -0.172 - j0.414, 0, -0.172 + j0.414, 0,$

$-5.828 + j2.414\}$.

→ Decimation - in - Frequency Algorithm

The DIT algorithm is based on the decomposition of the DFT computation by forming smaller and smaller subsequences of the sequence $x(n)$. In DIF algorithm the output sequence $X(k)$ is divided into smaller and smaller subsequences.

Summary of steps for Radix-2 DIF-FFT Algorithm

1. The no. of input samples $N = 2^M$, where M is no. of stages
2. The input sequence is in natural order
3. The no. of stages in the flow graph is given by $M = \log_2 N$
4. Each stage consists of $\frac{N}{2}$ butterflies
5. Inputs / outputs for each butterfly are separated by 2^{M-m} samples, where m represents the stage index i.e for first stage $m=1$ and for second stage $m=2$ so on.
6. The no. of complex multiplications is given by $\frac{N}{2}\log_2 N$
7. The no. of complex additions is given by $N\log_2 N$

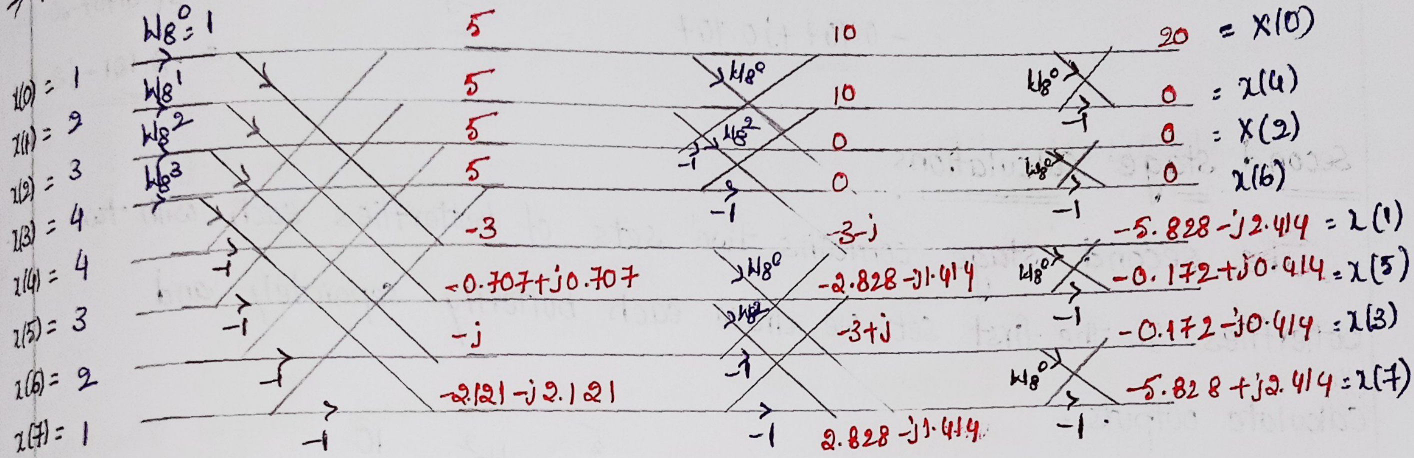
8. The twiddle factor exponents are a function of the stage index 'm' and is given by

$$k = \frac{Nt}{2^{M-m+1}}, \quad t = 0, 1, 2 \ldots 2^{M-m} - 1$$

9. The no. of sets or sections of butterflies in each stage is given by the formula 2^{m-1}

10. The exponent repeat factor (ERF) which is the no. of times the exponent sequence associated with 'm' repeated is given by 2^{m-1}

→ Find the DFT of a sequence $x(n) = \{1,2,3,4,4,3,2,1\}$ using DIF FFT algorithm.



$$W_8^0 = e^{\frac{-j2\pi 0}{N}} = 1$$

$$W_8^1 = e^{\frac{-j2\pi k}{N}} = e^{\frac{-j2\pi}{8}} = e^{-j\pi/4} = \cos\frac{\pi}{4} - j\sin\frac{\pi}{4} = 0.707 - j0.707$$

$$W_8^2 = e^{\frac{-j2\pi 2}{8}} = e^{\frac{-j\pi}{2}} = \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} = -j$$

$$W_8^3 = e^{\frac{-j2\pi 3}{8}} = e^{-j3\pi/4} = \cos\frac{3\pi}{4} - j\sin\frac{3\pi}{4} = -0.707 - j0.707.$$

In stage 1

$x(0) \longrightarrow 1+4 = 5$

$x(1) \longrightarrow 2 + 3 = 5$

$x(2) \rightarrow 3+2 = 5$

$x(3) \rightarrow 4+1 = 5$

$x(4) \rightarrow -4+1 = -3$

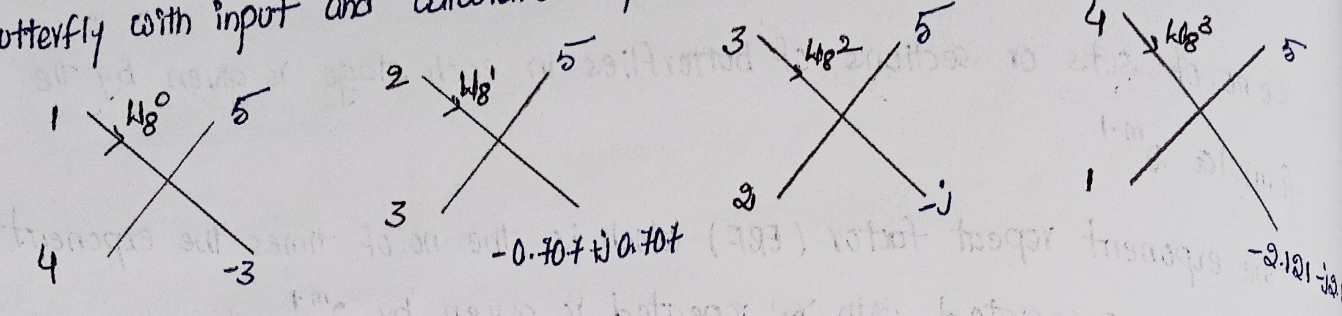
$x(5) \rightarrow -3+2(W_8^1)$

$= -3+2(0.707 - j0.707)$

$= -3 + 1.414 - j0.707$

First stage calculations.

The first stage contains a set of four butterflies. We draw individual butterfly with input and calculate outputs as below



$1+4=\underline{5}$

$(1-4)W_8^0 = \underline{-3}$

$2+3 = \underline{5}$

$(2-3)W_8^1 = -W_8^1$

$= \underline{-0.707+j0.707}$

$3+2=\underline{5}$

$(3-2)W_8^2 = W_8^2$

$= \underline{-j}$

$4+1=\underline{5}$

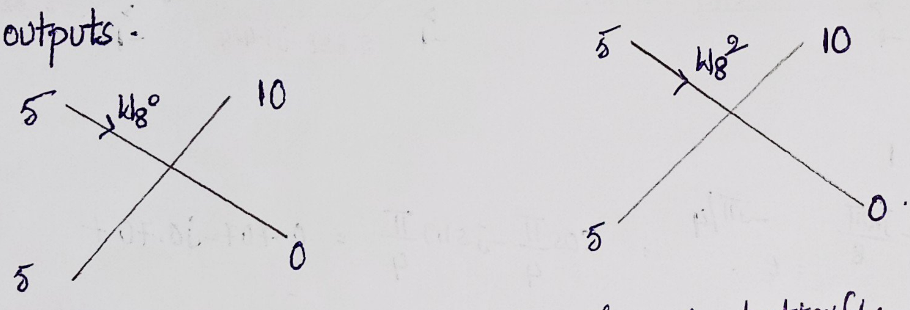
$(4-1)W_8^3 = 3W_8^3$

$3(-0.707-j0.707)$

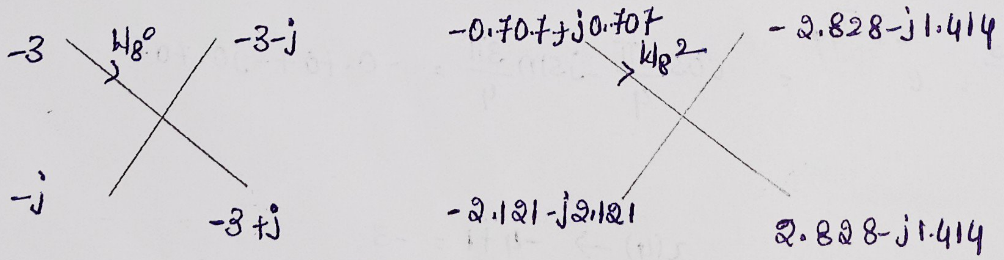
$= \underline{-2.121-j2.121}$

Second stage calculations.

The second stage contains two sets of butterflies each with two butterflies. In the first set we draw each butterfly separately and calculate outputs.



Similarily for second set the outputs of each butterfly can be calculate as below



$-0.707+j0.707-2.121-j2.121 = -2.828-j1.414$

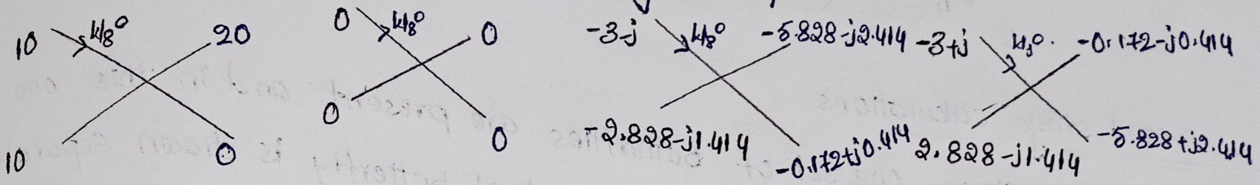
$(-0.707+j0.707+2.121+j2.121)W_8^2$

$(1.414+j2.828)(-j) = j1.414+2.828$

$= +2.828-j1.414$

Third stage calculations

The last stage contains four butterflies. The output of each butterfly can be obtain by adding and subtracting the inputs



$-3-j-2.828-j1.414$ $\qquad$ $-3+j+2.828-j1.414$

$-5.828-j2.414$ $\qquad$ $-0.172-j0.414$

$[(-3-j)-(-2.828-j1.414)]W_8^0$ $\qquad$ $-3+j-2.828+j1.414$

$-3-j+2.828+j1.414$ $\qquad$ $-5.828+j2.414$

$-0.172+j0.414$

$\therefore X(k)=\{20, -5.828-j2.414, 0, -0.172-j0.414, 0, -0.172+j0.414, 0, -5.828+j2.414\}$.

→ Compute 4-point DFT of a sequence $x(n)=\{0,1,2,3\}$ using DIT, DIF algorithm.

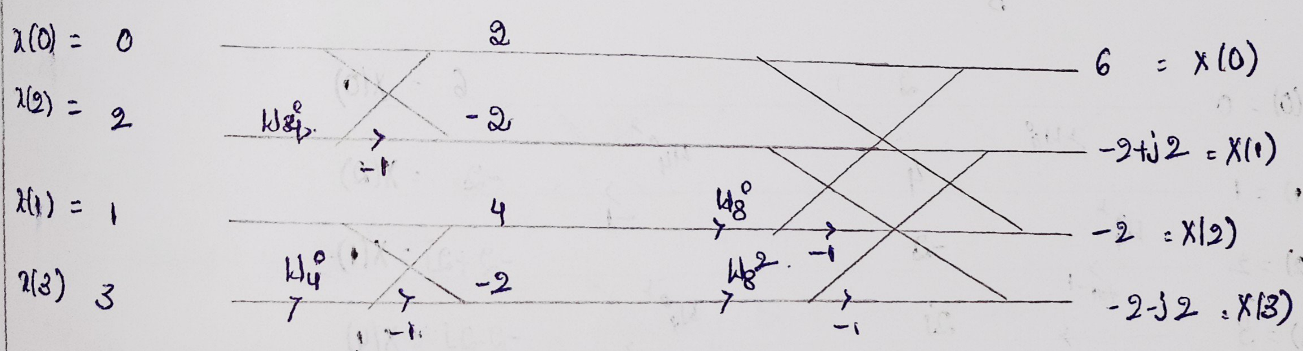
DIT algorithm

Sol: The given sequence $N: x(n)=\{0,1,2,3\}$.

No. of stages $M = \log_2 N = \log_2 4 = 2$.

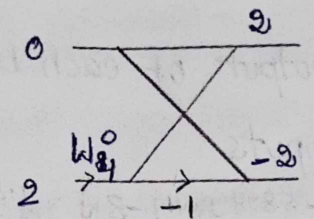
The twiddle factors are $W_4^0 = e^{\frac{-j2\pi 0}{4}} = e^0 = 1$

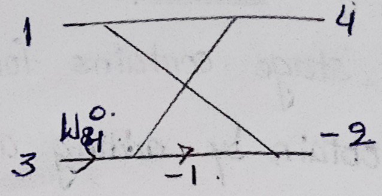
$W_4^1 = e^{\frac{-j2\pi}{4}} = e^{-j\pi/2} = \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} = -j$



First stage of calculations

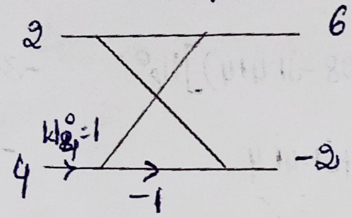
In this stage two butterflies are present. The input and output calculations are below.

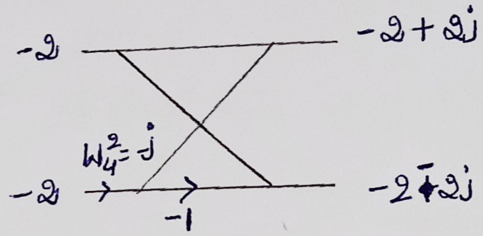




Second stage calculations

In this stage one set of butterflies are present, and in this one set two butterflies are present. The individual butterfly is drawn separately to calculate the outputs.





$-2+2j$

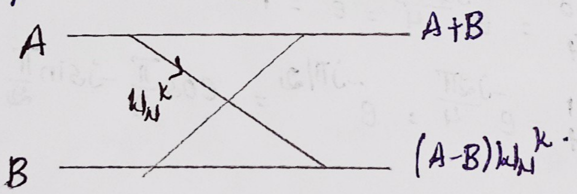
$\therefore X(K) = \{6, -2+2j, -2, -2-2j\}$.

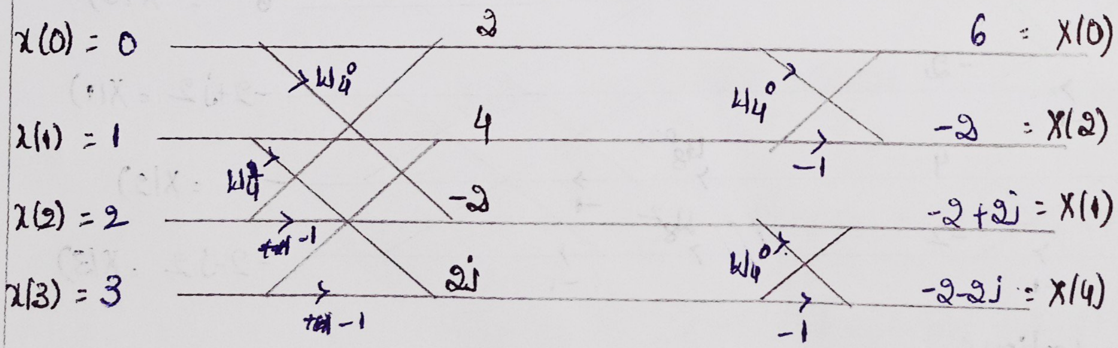
$(-2)-(-2)(-j) = -2-2j$

Input	stage I (S_1)	output
0	$0+2 = 2$	$2+4 = 6$
2	$0-2 = -2$	$-2+(-j)(-2) = -2+2j$
1	$1+3 = 4$	$2-4 = -2$
3	$1-3 = -2$	$-2-(-j)(-2) = -2-2j$

→ **DIF Algorithm**

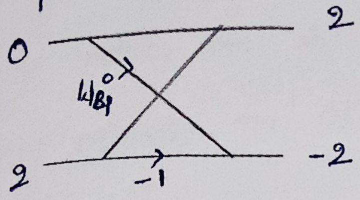
The butterfly operation is given by

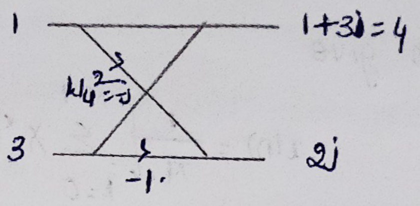




The first stage of calculations

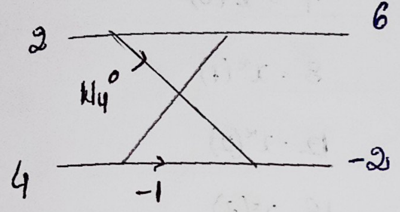
The first stage contains one set of two butterflies. The individual butterfly is drawn separately to calculate the outputs.

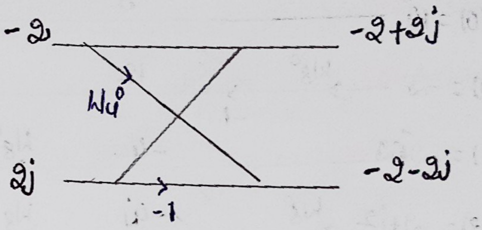




The second stage calculations

The second stage contains two butterflies. The outputs can be obtained by adding and subtracting inputs.





Input	S_1	Output
0	0+2 = 2	2+4 = 6
1	0-2 = -2	2-4 = -2
2	1+3 = 4	-2+2j
3	(1-3)(-j) = 2j	-2-2j

$\therefore X(k) = \{6, -2+2j, -2, -2-2j\}$

IDFT using FFT Algorithm

FFT algorithms can be used to compute an inverse DFT without any change in the algorithm. The inverse DFT of an N-point sequence $X(k)$, $k = 0, 1, \ldots N-1$ is defined as

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, W^{-nk}$$

where $W = e^{-j2\pi/N}$

Take complex conjugate and multiply by 'N', we obtain

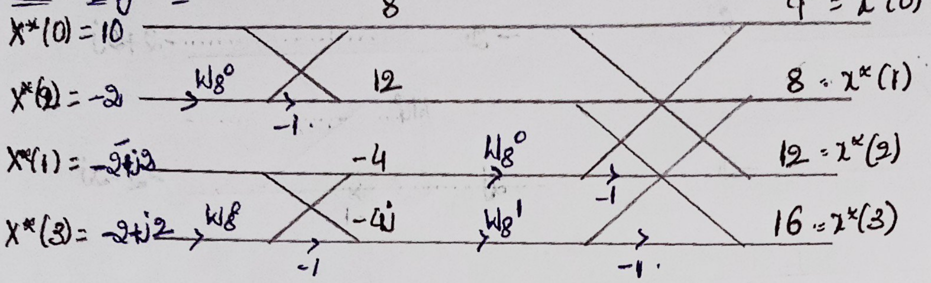
$$N x^*(n) = \sum_{k=0}^{N-1} X^*(k)\, W^{nk}$$

The right hand side of above equation is DFT of the sequence $X^*(k)$ and may be computed using any FFT algorithm. The desired output sequence $x(n)$ and then be found by complex conjugating the DFT and dividing by 'N' to give

$$x(n) = \frac{1}{N}\left[\sum_{k=0}^{N-1} X^*(k) W_N^{nk}\right]^*$$

→ Find the IDFT of the sequence $X(k) = \{10, -2+j2, -2, -2-j2\}$ using DIT algorithm.

Sol. DIT FFT algorithm



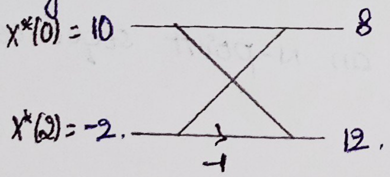
No. of samples $= N = 4$.

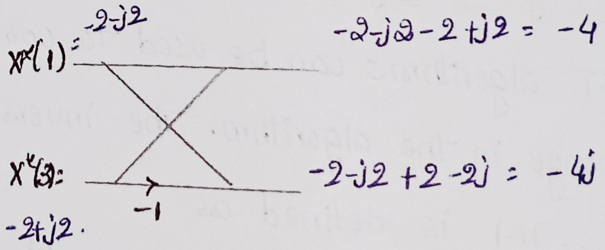
No. of stages $= M = \log_2 N = \log_2 4 = 2$.

The twiddle factors $W_4^0 = 1$; $W_4^1 = e^{\frac{-j2\pi \cdot 1}{4}} = e^{-j\pi/2} = \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} = -j$

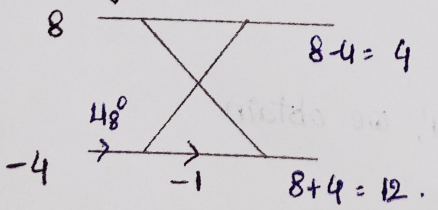
The first stage of calculations

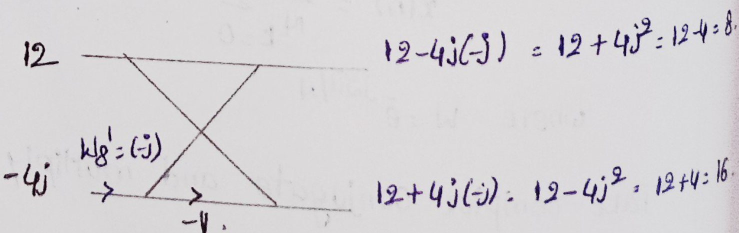
The first stage contain two butterflies. The input and output calculations are given below.





Second stage of calculations





The output $Nx^*(n)$ is normal order $\Rightarrow Nx^*(n) = \{4, 8, 12, 16\}$

Therefore $x(n) = \{1, 2, 3, 4\}$

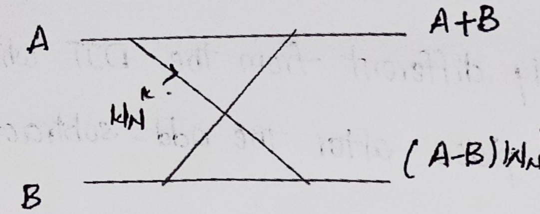
Find the IDFT of the sequence $X(K) = \{10, -2+j2, -2, -2-j2\}$ using DIF algorithm

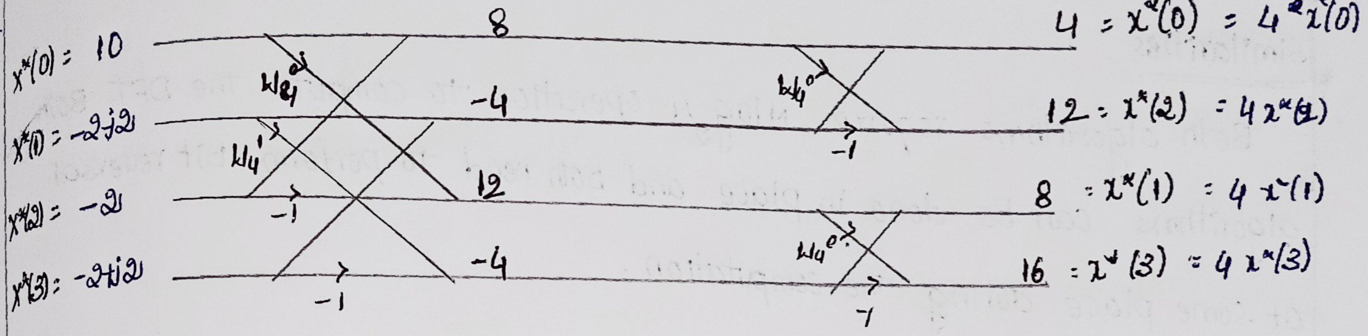
The given sequence $X(K) = \{10, -2+j2, -2, -2-j2\}$ = 4

No. of stages $M = \log_2 N = \log_2 4 = 2$

The twiddle factors are $W_4^0 = 1$; $W_4^1 = e^{\frac{-j2\pi}{4}} = e^{-j\pi/2} = -j$

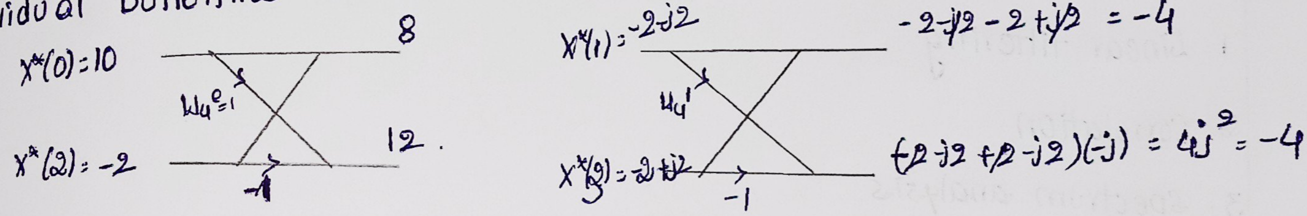
The butterfly operation for DIFFFT algorithm





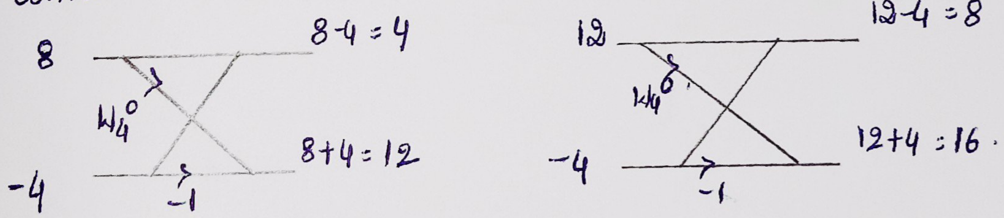
The first stage calculations.

The first stage contains one set of butterfliers contains two butterflies. The individual butterflies are shown below



Second stage calculations.

It contains two butterflies. The input and output are shown below



The output $Nx^*(n)$ is normal order $\Rightarrow Nx^*(n) = 4\{1,2,3,4\}$.

Therefore $x(n) = \{1,2,3,4\}$

→ Differences and similarities between DIT and DIF algorithms

Differences

1. For decimation-in-time (DIT), the input is bit-reversed while the output is in natural order. Whereas, for decimation-in-frequency the input is in natural order while the output is bit reversed order.

2. The DIF butterfly is slightly different from the DIT where in DIF the complex multiplication takes place after the add-subtract operation.

Similarities

Both algorithms require $N\log_2 N$ operation to compute the DFT. Both algorithms can be done in place and both need to perform bit reversal at some place during the computation.

→ Applications of FFT algorithms

1. Linear filtering
2. Correlation
3. Spectrum analysis